Introduction to Theory of Mesoscopic Systems

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Introduction
Einstein’s Miraculous Year - 1905

Six papers:


3. Brownian Motion. Received by Annalen der Physik May 11.

4, 5. The two papers on special relativity. Received June 30 and September 27

6. Second paper on Brownian motion. Received December 19.
Einstein’s Miraculous Year - 1905

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Diffusion and Brownian Motion:


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Q: Are these papers indeed important enough to stay in the same line with the relativity and photons. Why?
Einstein’s Miraculous Year - 1905

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Nobel Prize

By far the largest number of citations
Robert Brown (1773-1858)

The instrument with which Robert Brown studied Brownian Motion and which he used in his work on identifying the nucleus of the living cell. This instrument is preserved at the Linnean Society in London.
Brownian Motion - history

Robert Brown, *Phil. Mag.* 4, 161 (1828); 6, 161 (1829)

*Random motion* of particles suspended in water (“dust or soot deposited on all bodies in such quantities, especially in London”)

Action of water molecules pushing against the suspended object?

Giovanni Cantoni (Pavia). *N.Cimento*, 27, 156 (1867).
The Nobel Prize in Physics 1926

"for his work on the discontinuous structure of matter, and especially for his discovery of sedimentation equilibrium"

Jean Baptiste Perrin
France
b. 1870
d. 1942

... measurements on the Brownian movement showed that Einstein's theory was in perfect agreement with reality. Through these measurements a new determination of Avogadro's number was obtained.

The Nobel Prize in Physics 1926
From the Presentation Speech by Professor C.W. Oseen, member of the Nobel Committee for Physics of The Royal Swedish Academy of Sciences on December 10, 1926
Brownian Motion - history

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Random motion of particles suspended in water ("dust or soot deposited on all bodies in such quantities, especially in London")

Action of water molecules pushing against the suspend object

Problems:

1. Each molecules is too light to change the momentum of the suspended particle.
2. Does Brownian motion violate the second law of thermodynamics?

“We see under our eyes now motion transformed into heat by friction, now heat changes inversely into motion. This is contrary to Carnot’s principle.”

Problems:

1. Each molecule is too light to change the momentum of the suspended particle.
2. Does Brownian motion violate the second law of thermodynamics?
3. Do molecules exist as real objects and are the laws of mechanics applicable to them?
Kinetic theory

\[ S = k \log W + \text{const} \]

entropy

probability

Ludwig Boltzmann
1844 - 1906
Kinetic theory

\[ S = k \log W + \text{const} \]

\[ k \] is Boltzmann constant

entropy

probability

Ludwig Boltzmann
1844 - 1906

Max Planck
1858 - 1947

\[ \rho(\nu, T) = \frac{8\pi \hbar \nu^3}{c^3 \left[ \exp \left( \frac{\hbar \nu}{kT} \right) - 1 \right]} \]
From Micro to Macro

\[ S = k \log W + \text{const} \]

From Macro to Micro
“It is of great importance since it permits exact computation of Avogadro number ... . The great significance as a matter of principle is, however ... that one sees directly under the microscope part of the heat energy in the form of mechanical energy.”

Einstein, 1915
Brownian Motion - history

Einstein was not the first to:

1. Attribute the Brownian motion to the action of water molecules pushing against the suspended object
2. Write down the diffusion equation
3. Saved Second law of Thermodynamics

L. Szilard, Z. Phys, 53, 840(1929)
Einstein was not the first to:

1. Attribute the Brownian motion to the action of water molecules pushing against the suspended object
2. Write down the diffusion equation

Einstein was the first to:

1. Apply the diffusion equation to the probability
2. Derive the diffusion equation from the assumption that the process is markovian (before Markov) and take into account nonmarkovian effects
3. Derived the relation between diffusion const and viscosity (conductivity), i.e., connected fluctuations with dissipation

By studying large molecules in solutions sugar in water or suspended particles Einstein made molecules visible
Diffusion Equation

\[ \frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0 \]

Diffusion constant

Einstein-Sutherland Relation for electric conductivity \( \sigma \)

\[ \sigma = e^2 D \nu \quad \nu \equiv \frac{dn}{d\mu} \]

If electrons would be degenerate and form a classical ideal gas

\[ \nu = \frac{1}{Tn_{tot}} \]

William Sutherland (1859-1911)
Einstein-Sutherland Relation for electric conductivity $\sigma$

\[
\sigma dneD Edx = \text{No current}
\]

$\mu = \text{Density of electrons}$

$n = n(\mu)$

\[
\frac{dn}{dx} = \frac{dn}{d\mu} \frac{d\mu}{dx} = eE \frac{dn}{d\mu}
\]

$\mu = \text{Chemical potential}$

$eD \frac{dn}{dx} = \sigma E$

$\mu = \text{Electric field}$
Einstein-Sutherland Relation for electric conductivity $\sigma$

\[ n = n(\mu) \quad \frac{dn}{dx} = \frac{dn}{d\mu} \frac{d\mu}{dx} = eE \frac{dn}{d\mu} \]

No current
Density of electrons
Chemical potential
Electric field

\[ eD \frac{dn}{dx} = \sigma E \]

Conductivity

\[ \sigma = e^2 D \nu \quad \nu \equiv \frac{dn}{d\mu} \]

Density of states
Lessons from the Einstein’s work:

- **Universality**: the equation is valid as long as the process is markovian

- Can be applied to the **probability** and thus describes both fluctuations and dissipation

- There is a universal relation between the diffusion constant and the viscosity

- Studies of the diffusion processes brings information about micro scales.
What is a Mesoscopic System?

- Statistical description
- Can be effected by a microscopic system and the effect can be macroscopically detected

**Meso can serve as a microscope to study micro**

**Brownian particle was the first mesoscopic device in use**
Brownian particle was the first mesoscopic device in use

First paper on Quantum Theory of Solid State (Specific heat)
Annalen der Physik, 22, 180, 800 (1907)

First paper on Mesoscopic Physics
Annalen der Physik, 17, 549 (1905)
Finite size quantum physical systems

Atoms
Nuclei
Molecules

}\} Quantum Dots
1. Disorder ($\times$ – impurities)

2. Complex geometry

3. $e$–$e$ interactions
1. Disorder ($\times - \text{impurities}$)

2. Complex geometry

3. $\text{e-e interactions}$ for a while

**Realizations:**

- Metallic clusters
- Gate determined confinement in 2D gases (e.g. $\text{GaAs/AlGaAs}$)
- Carbon nanotubes
1. Disorder ($\times - \text{impurities}$)

2. Complex geometry

How to deal with disorder?

- Solve the Shrödinger equation exactly
- Start with plane waves, introduce the mean free path, and ... 

How to take quantum interference into account?
Lesson 1:

Beyond Markov chains:

Anderson Localization

and

Magnetoresistance
Quantum particle in random quenched potential
Absence of Diffusion in Certain Random Lattices

P. W. Anderson

Bell Telephone Laboratories, Murray Hill, New Jersey

(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.
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Anderson Model

- Lattice - tight binding model
- Onsite energies $\varepsilon_i$ - random
- Hopping matrix elements $I_{ij}$

$-W < \varepsilon_i < W$
uniformly distributed

$I_{ij} = \begin{cases} I & \text{i and j are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$

Anderson Transition

$I < I_c$
Insulator
All eigenstates are localized
Localization length $\xi$

$I > I_c$
Metal
There appear states extended all over the whole system
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhabkar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

$f = 3.04 \text{ GHz}$

$f = 7.33 \text{ GHz}$

Anderson Insulator  Anderson Metal
Classical particle in a random potential

1 particle - random walk
Density of the particles $\rho$
Density fluctuations $\rho(r,t)$ at a given point in space $r$ and time $t$.

$$\frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0$$

**Diffusion Equation**

$D$ - Diffusion constant

$$D = \frac{l^2}{d\tau}$$

$l$  mean free path
$\tau$  mean free time
$d$  # of dimensions
Einstein - Sutherland Relation for electric conductivity $\sigma$

$$\sigma = e^2 D \nu \quad \nu \equiv \frac{dn}{d\mu}$$

Conductance

$$G = \sigma L^{d-2}$$

for a cubic sample of the size $L$

$$g(L) = \frac{hD/L^2}{1/\nu L^d} = \frac{\text{Thouless energy}}{\text{mean level spacing}}$$

Dimensionless Thouless conductance

$$G = \frac{e^2}{h} \left( \nu L^d \right) \frac{Dh}{L^2} \frac{g(L)}{}$$
1. Mean level spacing

\[ \delta_1 = \frac{1}{\nu \times L^d} \]

$L$ is the system size; $d$ is the number of dimensions

2. Thouless energy

\[ E_T = \frac{hD}{L^2} \]

$E_T$ has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems)

\[ g = E_T / \delta_1 \]

$g$ is the Thouless conductance

\[ g = \frac{Gh}{e^2} \]
Scaling theory of Localization

(Abrahams, Anderson, Licciardello and Ramakrishnan 1979)

\[ g = \frac{E_T}{\delta_1} \]

Dimensionless Thouless conductance

\[ g = \frac{\mathcal{G}h}{e^2} \]

\[ L = 2L = 4L = 8L \ldots \]

without quantum corrections

\[ E_T \propto L^{-2} \quad \delta_1 \propto L^{-d} \]

\[ \frac{d\left(\log g\right)}{d\left(\log L\right)} = \beta(g) \]
\[ \frac{d(\log g)}{d(\log L)} = \beta(g) \]

\( \beta \) – function is 

Universal, i.e., material independent

But

It depends on the global symmetries, e.g., it is different with and without \( T \)-invariance (in orthogonal and unitary ensembles)

Limits:

\[
g \gg 1 \quad g \propto L^{d-2} \quad \beta(g) = (d - 2) + O\left(\frac{1}{g}\right)
\]

\[
g \ll 1 \quad g \propto e^{-L/\xi} \quad \beta(g) \approx \log g < 0
\]

> 0 \quad d > 2

?? \quad d = 2

< 0 \quad d < 2
$\beta$ - function

$$\frac{d \log g}{d \log L} = \beta(g)$$

Metal – insulator transition in $3D$

All states are localized for $d=1,2$
Why diffusion description fails at large scales?

Questions:
• Why the scaling theory is correct?
• Why the corrections of the diffusion constant and conductance are negative?
Diffusion description fails at large scales

Why?

Einstein: there is no diffusion at too short scales – there is memory, i.e., the process is not marcovian.
Diffusion description fails at large scales
Why?

Einstein: there is no diffusion at too short scales - there is memory, i.e., the process is not markovian.

Why there is memory at large distances in quantum case?

Quantum corrections at large Thouless conductance - weak localization
Universal description
\[ \beta(g) = d - 2 + \frac{c_d}{g} \]

\[ g(L) = \sigma_c L^{d-2} - \frac{c_d}{d-2} - c_2 \log\left(\frac{L}{l}\right) \]

\[ \frac{d \log g}{d \log L} = \beta(g) \]