Thanks to collaborations with:

I. Beloborodov, A. Lopatin, and K. Efetov
Plan

Introduction: general features of granular conductors

1. Granular metals
   • Metallic region
     Granular metals and quantum dots
     Universal and Fermi-liquid behavior
     Weak localization effects
   • Insulating region - hopping conductance

2. Granular superconductors
   • Metallic region
   • Insulating region - hopping conductance
   • Transition - negative magnetoresistance
New generation of granular materials: Arrays of nanocrystals/quantum dots

There is a large, increasingly important class of materials, including quench-condensed metal films and arrays of metallic or semiconducting quantum dots, which consist of weakly-coupled nanometer-scale islands.

Quantum dots are nanoscale particles made up of hundreds to thousands of atoms that (artificially designed arrays of nanocrystals or quantum dots, tiny crystals ranging in size from 2 to 30 nanometers), in many of their characteristics, behave like a single gigantic atom. The remarkable optical and transport properties of quantum dots—particularly the ease of customizing those properties by adjusting the size or composition of the dots—assures them a fundamental role in the emerging field of molecular electronics.
Due to small size, nanocrystals have properties between those of molecules and bulk materials, providing the opportunity to uniquely modify the material via altering composition and size of the particles. Controlling these parameters during synthesis allows tailoring transport, optical, and magnetic properties of the arrays for specific applications ranging from the new generation fluorophores, optical switching, optical transistors, optical computing or telecommunications packet switching, to solar batteries. Using quantum dot arrays opens a new route for making semiconductors for technological applications with an engineered electronic structure resulting in a more effective material.
A new class of water-soluble quantum dots made from small numbers of gold atoms could be the basis for a new biological labeling system with narrower excitation spectra, smaller particle size and fluorescence comparable to systems based on semiconductor quantum dots.

Fluorescence is shown from solutions of small gold nanoclusters dissolved in water. These nanoclusters behave like multi-electron artificial atoms, emitting at discrete wavelengths in the visible and infrared with the wavelength increasing with the size of the cluster. Shown from left to right are emissions from gold nanoclusters containing 5, 8 and 13 atoms.
Solar radiation is a plentiful and clean source of power. However, due to the high cost of electrical conversion on a per Watt basis, conventional solar cells have not been exploited to their full potential. The cost per watt can be reduced by increasing the efficiency of solar cells and by reducing their cost to manufacture. The high price for solar cells is largely due to the use of expensive substrate materials and costly microfabrication processing.
It is expected that CdSe and PbSe semiconductor nanocrystals that are ideally suited as solar cell material due to the ability to tune the electronic bandgap and, hence, optimize a solar cell for maximum efficiency, will replace conventional solar cells, which due to the high cost of electrical conversion on a per watt basis prevent exploiting solar radiation as inexhaustible and clean source of power to its full potential.
Nanomagnetism: Ferromagnetic quantum dots produced by this technique have exhibited record coercivity

Single electronics: Semiconductor quantum dots have shown room temperature Coulomb blockade and 50-nm diameter CdS and ZnSe nanowires have shown Coulomb staircase at room temperature. These systems can be used in novel computing architectures based on collective computational paradigms.

Quantum dot non-volatile memory: a novel electronic bistability in short nanowires
bulk material

bandgap

nanocrystal

\( \delta \)
nanocrystal
The key characteristic: tunneling conductance

The tunneling conductance is measured in the units of the quantum conductance $e^2 / 2\hbar$

$g_T \gg 1$ metallic transport properties

$g_T \ll 1$ insulating behavior
Granular metals: experiment

A. Gerber et al. PRL 78. 4277 (1997)

Samples with larger coupling showed the behavior as

\[ R \propto T^{-\alpha} \approx 1 - \alpha \ln T, \quad \alpha \approx 0.117 \]

The samples are 3D, and the experiment was carried out in high magnetic field, excluding thus the explanation of log as the weak localization behavior.

FIG. 3. Resistance of sample 3 as a function of temperature on a log-log scale, as measured at (zero) (×) and 100 kOe field (open circles). Open circles indicate resistance measured with a constant dc current \( I = 10^{-5} \) A. Solid squares are zero bias resistances approximated from \( I-V \) measurements. Sample 3 room temperature resistance is 500 \( \Omega \).
Granular metals: experiment.


NbN in the insulating substrate

![Graph showing the temperature dependence of resistance](image1)

A. Gerber et al, PRL 78, 4277 (1997)

Samples (Al-Ge) with the high room temperature resistivity (weak tunneling) showed an exponential growth of resistivity as function of temperature.

![Graph showing the relationship between temperature and log of resistivity](image2)

\[ \sigma \propto \exp \left[ -\left( \frac{T_0}{T} \right)^{1/2} \right] \]
Granular metallic arrays

Array of metallic nanoparticles in an insulating substrate:

Disorder:
Motion of electrons inside the grains is diffusive

The electron spends some time inside the grain before it tunnels into a neighboring grain

Electron diffusion  Tunneling junction

Coulomb interaction

Charging energy \( E_c = e^2 n^2 / C \)

Coulomb blockade.
parameters characterizing granular sample I:

**Homogeneous metal**

- diffusive electron motion

**Granular metal**

*Energy scales:*

- $E_F$ - Fermi energy
- $\tau$ - mean free time
- $\delta$ - mean energy level spacing
- $E_c \approx e^2/a$ - Coulomb energy
- $t$ - tunneling matrix element
parameters characterizing a granular sample II:

**Tunneling conductance:** \( g_T = \frac{I}{V} = \frac{t^2}{\delta^2} \)

Quantum conductance: \( g_Q = \frac{e^2}{\hbar} \)

Corresponding resistance: \( R_Q = 4.05 \ k\Omega \)

Dimensionless tunneling conductance: \( g_T \rightarrow \frac{g_T}{g_Q} \)

Internal conductance of a single grain: \( g_o \)

**Coulomb energy:** \( E_C \approx \frac{e^2}{a} \)

Mean distance between the energy levels \( \delta \):

For realistic granules: \( E_C \gg \delta \)
\[
\frac{1}{\tau_T} = \frac{2\pi}{\hbar} \left\langle t^2 \right\rangle \nu_r(\varepsilon_F)
\]

\[
G = \frac{e^2 \nu_l(\varepsilon_F)}{\tau_T}
\]

\[
N = eV \nu_l(\varepsilon_F) \quad \Rightarrow \quad I = eN / \tau_T = e^2 \nu_l(\varepsilon_F) V / \tau_T
\]

\[
\sigma_0 \approx G a^{2-d} = \frac{e^2}{\pi \hbar} g_T a^{2-d}
\]
Coulomb blockade in a single grain.

**Single grain + metallic lead**

- **$g_T <\ll 1$** - Coulomb blockade regime - charge quantization
- **$g_T >\gg 1$** - Charge quantization effects are exponentially small

For a granular system one can expect the metal insulator transition at $g_T \sim 1$. 
Conductivity of a granular sample

1. Insulating regime (weak coupling between the grains $g_T << 1$)

Periodic granular array:
thermally activated hopping conductivity with the Coulomb gap - $E_c$

\[ \sigma \sim e^{-2E_c/T} \]

One would expect:

Typical experimental dependence: $\sigma \sim e^{-\Lambda/\sqrt{T}}$

We consider this regime later
Conductivity of a granular sample

2. Metallic regime (strong coupling between the grains $g_T >> 1$)

Q: What is the temperature dependence of the conductivity of a granular metallic sample?

Possible approaches:

1. **Model of incoherent electron tunneling**
   Temperature dependence of conductivity is controlled by electron tunneling between the neighboring grains

2. **Model of a disordered metal**
   Temperature dependence of conductivity is similar to that of the disordered metals

Important energy scale:
$$\Gamma \sim g_T \delta$$
Smearing of the energy levels due to inter-grain electron tunneling

$$\Gamma = \frac{\hbar}{\tau_a}$$

Depending on the relation between the temperature $T$ and $\Gamma$ both regimes are possible:

1. $T \gg \Gamma$ - "high temperature" incoherent regime – granularity is important
2. $T \ll \Gamma$ - "low temperature regime" similar to homogeneous disordered metals
High temperature \((T > g_T \delta)\) conductivity

Model of incoherent electron intergranular tunneling.

Drude conductivity (no interaction corrections) \(\sigma_0 = \frac{e^2}{\pi \hbar} g_T a^{2-d} \) \((a \text{ is the grain size})\)

Simplified theoretical approach – effective action written in terms of the phase variables

Conductivity: \(\sigma = \sigma_0 + \delta \sigma\) \(\delta \sigma\) - interaction correction

Perturbation theory in \(1/g_T\) results in the correction to conductivity

\[
\frac{\delta \sigma}{\sigma_0} = - \frac{1}{2\pi d g_T} \ln \left( \frac{g_T E_C}{T} \right)
\]

\text{Efetov, Tschersich (2003)}

Can be understood as the renormalization of the tunneling conductance between the neighboring grains

\[
\tilde{g}_T = g_T - \frac{1}{2\pi d} \ln \left( \frac{g_T E_C}{T} \right), \quad \text{valid as long as } \tilde{g}_T \gg 1.
\]

Conductivity depends logarithmically on temperature for in all dimensions!
Second order in tunneling

$$S = S_c + S_t$$

Charging part

$$S_c = \frac{1}{2e^2} \sum_{ij} \int_0^\beta d\tau C_{ij} \frac{d\phi_i(\tau)}{d\tau} \frac{d\phi_j(\tau)}{d\tau}$$

Tunneling part

$$S_t = \pi g_T \sum_{ij} \int_0^\beta d\tau d\tau' G^2(\tau - \tau') \sin^2 \left( \frac{\phi_{ij}(\tau) - \phi_{ij}(\tau')}{2} \right)$$

$$G(\tau) = \frac{T}{\sin(\pi T \tau)}$$
Low temperature conductivity: \( T < g_T \delta \)

\[
\sigma = \sigma_0 + \delta \sigma \quad \text{where} \quad \delta \sigma = \delta \sigma_1 + \delta \sigma_2
\]

\( \delta \sigma_1 \) comes from the large energies, \( \varepsilon > g_T \delta \), where the granular structure of the array dominates the physics. The fact that this correction is independent of the dimensionality \( d \) means that the tunneling of electrons is incoherent.

\( \delta \sigma_2 \) is similar to AA correction for homogeneously disordered metals. This is the contribution from the low energies, \( \varepsilon < g_T \delta \), the behavior is dominated by coherent electron motion on large scales.
\[
\frac{\delta \sigma_1}{\sigma_0} = -\frac{1}{2\pi d g_T} \text{Im} \sum_q \int d\omega \gamma(\omega) \varepsilon_q \tilde{V}(\omega, q)
\]

\[
\gamma(\omega) = \frac{d}{d\omega} \omega \coth \frac{\omega}{2T}, \quad \varepsilon_q = 2g_T \sum_a (1 - \cos qa)
\]

\[
\tilde{V}(\omega, q) = \frac{2E_C(q)}{(\varepsilon_q \delta - i\omega)[4\varepsilon_q E_C(q) - i\omega]}
\]

\[
\frac{\delta \sigma_1}{\sigma_0} = -\frac{1}{2\pi d g_T} \ln \left[ \frac{g_T E_C}{\max(T, g_T \delta)} \right]
\]

Efetov & Tschersich 2003
\[ \delta \sigma_2 = \quad \]
Final result for conductivity: \( \sigma = \sigma_0 + \delta \sigma \)

1) High temperatures: \( T > g_T \delta \)

\[
\frac{\delta \sigma}{\sigma_0} = -\frac{1}{2\pi d g_T} \ln \left[ \frac{g_T E_C}{T} \right]
\]

2) Low temperatures: \( T < g_T \delta \)

\[
\frac{\delta \sigma}{\sigma_0} = -\frac{1}{2\pi d g_T} \ln \left[ \frac{E_C}{\delta} \right] + \begin{cases} 
\frac{\alpha}{12\pi^2 g_T} \sqrt{\frac{T}{g_T \delta}}, & D = 3 \\
-\frac{1}{4\pi^2 g_T} \ln \frac{g_T \delta}{T}, & D = 2 \\
-\frac{\beta}{4\pi} \sqrt{\frac{\delta}{T g_T}}, & D = 1
\end{cases}
\]
The low temperature dependence of the conductivity of granular metals coincides exactly with the corresponding result for the conductivity of the homogeneously disordered samples. A question immediately arises: is it a coincidence that two different physical systems exhibit identical low temperature transport behaviors, or there is an underlying deep connection between the two? Furthermore, do all the other physical quantities (specific heat, tunneling density of states, etc.) possess the same universality?

One has to seek for a universal description in terms of the appropriate large scale degrees of freedom that characterize disordered conductors.
We construct such a universal description of low temperature physical properties of granular metals building on the $\sigma$-model. Our approach applies at temperatures $T < g_T\delta$. The energy scale $g_T\delta$ appears naturally as the upper energy cutoff of the effective model, since it is the mean time for the electron to escape from the granule $\hbar/g_T\delta$.

\[ \hat{H} = \hat{H}_0 + \hat{H}_{int} + \sum_{ij} \hat{\psi}^\dagger(r_i) t_{ij} \hat{\psi}(r_j) \]

where $t_{ij}$ is the tunneling matrix elements ($t_{ij} = t_{ji}$)

\[ \hat{H}_0 = \sum_i \int d^3 r_i \hat{\psi}_i^\dagger \left[ \frac{\hat{p}^2}{2m} - \mu + u(r_i) \right] \hat{\psi}_i \]

\[ \hat{H}_{int} = \frac{e^2}{2} \sum_{ij} \hat{n}_i C_{ij}^{-1} \hat{n}_j \]

We decouple the Coulomb interaction term using the auxiliary fields $V$, average over disorder introducing the $Q$–matrix field, and expand around the diffusive saddle point.
Field $V$ is a vector in the frequency and replica spaces and the corresponding contraction is assumed: $V^*V = \sum_{\omega_n, \alpha} V^*_{\omega_n, \alpha} V_{\omega_n, \alpha}$

The $Q$-matrices are matrices in the Matsubara, spin, and replica spaces

$$Q = Q^{a,b}_{\omega_{n1}, \omega_{n2}; \alpha, \beta} \quad Q^2 = 1$$

To derive the low energy model that provides the universal description of the low temperature phase, $T < g_T \delta$, one applies the renormalization group technique to the effective action.
RG equation:

\[ \frac{d g_T}{d \ln \Lambda} = -\frac{1}{2\pi d} \]

\[ g_T = g_T^{(0)} - \frac{1}{2\pi d} \ln \left( \frac{E_C}{\delta} \right) \]

\[ g_T^c = \frac{1}{6\pi} \ln \frac{E_C}{\delta} \]

INSULATOR \hspace{2cm} METAL

\[ g_T^c \]

\( g_T \) (high temperature conductance)

For \( E_C \approx 100K \) and \( \delta \approx 1K \)

\[ g_T^c \approx 0.24 \] (!)
At energies lower than $g_T \delta$ the physics is dominated by the distances that are much larger than the size of a single grain.

\[
S = -\frac{\pi}{2\delta} \int \text{Tr} \left[ (\hat{\varepsilon} + V)Q - \frac{D}{4}(\nabla Q)^2 \right] \frac{d\mathbf{r}}{a^d} \\
+ \frac{1}{T} \int \frac{d\mathbf{r} d\mathbf{r}'}{a^{2d}} \text{Tr} \left[ V_r^* \frac{C_{rr'}}{2e^2} V_{r'} \right] D = g_T a^2 \delta
\]

Since this effective model operates with the $Q$–matrices which have only long-range degrees of freedom, it applies, with the appropriate charges and upon the high-energy renormalization, to any disordered metal, including a homogeneously disordered one.

\[
\sigma = 2e^2 D (a^d \delta)^{-1}
\]
\[ \delta \sigma_1 = -\sigma_0 \frac{1}{2\pi d g_T} \ln \left[ \frac{E_C}{\delta} \right] \]

\[ \sigma_0 = 2e^2 g_T a^{2-d} \]

\[ \delta \sigma_2 = \sigma_0 \begin{cases} 
\frac{\alpha}{12\pi^2 g_T} \sqrt{\frac{T}{g_T \delta}} \\
- \frac{1}{4\pi^2 g_T} \ln \frac{g_T \delta}{T} \\
- \frac{\beta}{4\pi} \sqrt{\frac{\delta}{T g_T}} 
\end{cases} \]

\[ d = 3, \quad \alpha \approx 1.83 \]
\[ d = 2, \quad \beta \approx 3.13 \]
\[ d = 1. \]
Weak localization corrections

\[ \delta \sigma_{WL} = -\frac{2}{\pi} e^2 g_T \int C(0, q) \frac{d^d q}{(2\pi)^d} \]

\[ C(0, q) \text{ is the Cooperon propagator} \]

\[ C(\omega, q) = (Dq^2 - i\omega)^{-1} \]

\[ \frac{\delta \sigma_{WL}}{\sigma_0} = -\frac{1}{4\pi^2 g_T} \ln \left( \frac{g_T^2 \delta}{T} \right) \]

films

\[ \frac{\delta \sigma_{WL}}{\sigma_0} = -\frac{1}{2\pi g_T} \left( \frac{g_T^2 \delta}{T} \right)^{1/3} \]

wires
Phase Diagram of Granular Metallic Systems


insulator

metal – insulator transition

activated conductivity

universal logarithmic corrections

\[ T > g_T \delta \]

\[ T < g_T \delta \]

\[ \delta \sigma \sigma_0 = - \frac{1}{2 \pi d g_T} \ln \left( \frac{E_C}{\delta} \right) + \begin{cases} \frac{\alpha}{12\pi^2 g_T} \sqrt{\frac{T}{g_T \delta}}, & D = 3 \\ - \frac{1}{4\pi^2 g_T} \ln \frac{g_T \delta}{T}, & D = 2 \\ - \frac{\beta}{4\pi} \sqrt{\frac{\delta}{T g_T}}, & D = 1 \end{cases} \]

\[ g_T \sim 1/R \]

\[ T = g_T \delta \]

\[ \delta - \text{mean level spacing} \]
Conductivity of a periodic granular sample in the insulating regime

Weak coupling between the grains: \( g_T \ll 1 \)

Periodic granular sample:
activation conductivity with the Coulomb gap, \( E_c \), as activation barrier

\[
\sigma \sim e^{-\Delta_M/T}, \quad \Delta_M = E_c, \quad g \ll 1
\]

Mott gap \( \Delta_m \) is reduced due to intergranular electron tunneling

Weak coupling: perturbation theory

\[
\Delta_M = E_c - \frac{2gz}{\pi} E_{eh} \ln 2, \quad g_z \ll 1
\]

\( E_{eh} \) - energy to create an electron-hole excitation

\( z \) – coordination number
Reduction of the Mott gap at stronger coupling. Mott transition

1. Self consistent nearest neighbor hopping approximation

Mott gap is suppressed exponentially

\[ \Delta_M = c g E_c e^{-\pi g z}, \quad g z \gg 1 \quad c \text{ – const} \]

2. The above result neglects electron motion on scales of many grains

Diffusive time \( \tau \sim D^{-1} r^2 \quad r \text{ – distance, } D \text{ – diffusive coefficient} \)

Diffusive processes are suppressed as long as \( \tau > \Delta_M^{-1} \).

\( D = g \delta a^2, \quad \delta \text{ - mean energy level spacing in a grain} \)

Taking: \( r \sim \) grain size \( a \)

\[ \Delta_M > g \delta \quad \Rightarrow \quad g < g_c = \frac{1}{\pi z} \ln(E_c/g\delta) \]

Mott transition at \( T=0 \)

\( g > g_c \text{ – metal} \)

\( g < g_c \text{ – insulator} \)
Experimental observations, earlier explanation attempts

Typical experimental dependence: \[ \sigma \sim e^{-A/T^p}, \quad p \approx 1/2 \]


Earlier attempts to explain the conductivity temperature dependence were based on

**ASCA phenomenological model:**

- Thickness of the insulating layer between two grains is proportional to grain sizes: \[ t \sim R. \]

- Coulomb energy: \[ E_c \sim e^2/R \]

- Tunneling probability: \[ P \sim e^{-2t/L} \text{ L – localization length of the insulating layer} \]

- Optimization of \[ e^{-e^2/RT-2t/L} \text{ under constraint } R \sim t \]

  results in \[ p=1/2 \] dependence
Irregular arrays, role of the electrostatic disorder.

Critique of ASCA model:  

1. Capacitance disorder cannot remove the Coulomb gap completely
   
   ASCA model cannot explain the observed behavior at low temperature

2. Recent experiments showed the $p=1/2$ law for periodic arrays.
   
   2d array of gold particles of size $\sim 5.5$ nm.
   Particle sizes are controlled within $5\%$ accuracy.

D. Yu, C. Wang, B. L. Wehrenberg, P. Guyot-Sionnest PRL 2004
T.B. Tran, et al, PRL 2005

ES law was also observed in the nanocrystal arrays of semiconducting quantum dots

Yakimov, et al, JETP Lett. 2003
Hopping conductivity: Random potential model

Two crucial ingredients of the hopping conductivity:

a) Finite density of states in the vicinity of the Fermi level.

b) Ability to hop on distances larger than a single grain size.

Capacitance fluctuations – not enough.

Model: random potential is applied on each grain

\[ H = \sum_i V_i \hat{n}_i + \sum_{ij} \hat{n}_i E_{ij}^c \hat{n}_j \]

\( \hat{n}_i \) - electron density on grain i  \( V_i \) - random potential on grain i

Random potential gives rise to the flat density of states. Coulomb correlations \( \rightarrow \) Efros-Shklovskii suppression of DOS.
Coulomb interactions and hopping conductivity in arrays of quantum dots

\[ H = \sum_i \mu_i n_i + \sum_{ij} n_i E_{ij}^c n_j \]

\[ E_i^\pm = E_i^c \pm \mu_i \]

This looks exactly like the impurity levels in doped semiconductors:

VRH?
Coulomb interactions.

In the presence of Coulomb interaction the site energies are renormalized by the surrounding sites.

Denoting the renormalized energies as $\tilde{\varepsilon}_i$, let us estimate the energy cost for an electron transfer from a filled, $i$, to an empty donor, $j$:

$$\Delta^j_i = \tilde{\varepsilon}_j - \tilde{\varepsilon}_i - \frac{e^2}{\kappa r_{ij}}$$

It follows that the states close to the Fermi level must be located at large distances.

$$g(\tilde{\varepsilon}) \propto \left( \frac{\kappa^2}{e^6} \right) \cdot \tilde{\varepsilon}^2$$
Very crude estimate

\[ \sigma \propto \exp\left(-\frac{r}{\xi}\right)\exp\left(-\frac{e^2}{\kappa r T}\right) \]

Optimizing with respect to \( r \), find:

\[ \sigma \propto \exp\left[-\left(\frac{T_{ES}}{T}\right)^{1/2}\right] \]

\[ T_{ES} = \frac{e^2}{\kappa \xi} \]
Sketch of derivation:

Consider a narrow band near the Fermi level

\[ \mu - \varepsilon/2 \leq \varepsilon_{ij} \leq \mu + \varepsilon/2, \quad \varepsilon_i \cdot \varepsilon_j < 0 \]

To be in the ground state we have to assume that \( \Delta_i^j > 0 \rightarrow r_{ij} > e^2/\kappa\varepsilon \)

Then the concentration is \( n(\varepsilon) \sim \varepsilon^3 \kappa^3/e^6 \sim \varepsilon g(\varepsilon) \)

As result, \( \zeta_{ES} = \left(\frac{T_{ES}}{T}\right)^{1/2} \), \( T_{ES} = \beta_{ES} \frac{e^2}{\kappa \alpha} \)

Thus:

\[ R_h \propto e^{\zeta_{ES}} \propto e^{(T_{ES}/T)^{1/2}} \quad \text{Efros-Shklovskii law} \]
Weak intergrain tunneling coupling $\rightarrow$ quantized charge $\rightarrow$ classical description

$$H = \sum_i V_i n_i + \sum_{ij} n_i E_{ij}^c n_j \quad n_i = 0, 1 - \text{classical electron charge}$$

The model is essentially equivalent to the one studied by Efros and Shklovskii.

**Coulomb gap: Efros-Shklovskii result**

$$\nu_g(\varepsilon) \sim \left(\tilde{\kappa}/e^2\right)^d \varepsilon^{d-1}$$

$\tilde{\kappa}$—effective dielectric constant

For granular metals ES result gives the density of the GROUND states.

Description in terms of the classical model is degenerate:
Many electron states within each grain correspond to the same charge.

**DOS:**

$$\nu(\varepsilon) \sim \nu_0 \left(\varepsilon \tilde{\kappa}/e^2\right)^d$$

In the Mott criterion for finding the hopping distance $r$ within the energy shell $\varepsilon$

$$r^d \int_0^\varepsilon d\varepsilon' \nu_g(\varepsilon') \sim 1$$

one has to use the density of GROUND states (DOGS)
arrays of dots: electrons have to tunnel through other granules

tunneling via virtual states of intermediate grains

Elastic cotunneling mechanism

\[ T < \sqrt{E_0^c\delta} \]

Inelastic cotunneling mechanism

\[ T > \sqrt{E_0^{ic}\delta} \]
\[ P_{el} \sim e^{-2s/\xi_{el}} \]

\[ \xi_{el} = \frac{2}{\ln \left( \bar{E} \pi/c \tilde{g} \delta \right)} \]

\[ \ln \bar{E} = \frac{1}{N} \sum_{k=1}^{N} \ln \tilde{E}_k \]

\[ \tilde{E}_k = 2 \left( \frac{1}{E_k^+} + \frac{1}{E_k^-} \right)^{-1} \]

hopping probability \( \propto \exp \left[ -\left( \frac{2s}{\xi_{el}} \right) - \left( \frac{e^2}{\kappa T a s} \right) \right] \)
Elastic cotunneling mechanism

Tunneling through a chain of grains

Model: Short range on-site interaction:

Electron (hole) excitation energies

\[ E^{\pm}_i = E^c_i \pm \mu_i \]

Tunneling probability is a product

\[ P_{el} = \delta(\xi_{N+1} - \xi_0) g_0 \prod_{k=1}^{N} P_k \]

of elementary probabilities

\[ P_k = \frac{g_k \delta_k}{\pi E_k} \quad \tilde{E}_k = 2 \left( \frac{1}{E^+_k} + \frac{1}{E^-_k} \right)^{-1} \]

\( g_k \) - conductance between k-th and k+1-st grains

In terms of geometrical averages along the tunneling path the probability is

\[ P_{el} = \bar{g}^{N+1} \left( \frac{\bar{\delta}}{\pi \bar{E}} \right)^N \delta(\xi_{N+1} - \xi_0) \]

\[ \ln \bar{E} = \frac{1}{N} \sum_{k=1}^{N} \ln \tilde{E}_k \]

\[ P_{el} \sim e^{-2s/\xi_{el}} \quad s - \text{distance along the path} \]

Effective localization length:

\[ \xi_{el} = \frac{2a}{\ln(\bar{E} \pi / \bar{g} \delta)} \]
Hopping conductivity in the regime of elastic cotunneling

Variable range hopping: Phonon assisted tunneling:

\[ I \sim e^{-2r/\xi_{el}} - \varepsilon/T \]

(Granular metals: electrons also contribute to the energy relaxation)

Hopping distance \( r \) within the energy shell \( \varepsilon \) is given by

\[ r^d \int_{0}^{\varepsilon} d\varepsilon' \nu_g(\varepsilon') \sim 1 \]

E.S. DOGS

\[ \nu_g(\varepsilon) \sim (\tilde{\kappa}/e^2)^d |\varepsilon|^{d-1} \rightarrow r \in \tilde{\kappa}/e^2 \sim 1 \]

Minimization results in the E.S. law:

\[ \sigma \sim e^{-(T_0/T)^{1/2}} \quad T_0 \sim e^2/\tilde{\kappa}\xi_{el} \quad \xi_{el} = \frac{2a}{\ln(E\pi\tilde{g}\delta)} \]

Nonlinear conductivity at strong electric fields

Hopping distance \( r \) within the energy shell \( \Delta \):

\[ e\varepsilon r \sim \Delta \]

\[ r \Delta \tilde{\kappa}/e^2 \sim 1 \]

\[ r \sim \sqrt{e/\tilde{\kappa}\varepsilon} \]

\[ j \sim j_0 e^{-(\varepsilon_0/\varepsilon)^{1/2}} \quad \varepsilon_0 \sim T_0/e\xi_{el} \]

Shklovskii 1973

\[ \varepsilon - \text{electric field} \]

\[ e\varepsilon r \]

\[ \Delta \]
Inelastic cotunneling: single granule

D. A. Averin and Yu. V. Nazarov, PRL (1990)
Cotunneling allows for charge transport through several junctions at a time by cooperative electron motion.

At low temperature the sequential tunneling is exponentially suppressed by the Coulomb blockade.

In this case, a higher-order tunneling process transferring electron charge coherently through two junctions can take place. The excess electron charge at the grain exists only virtually.

\[
\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} \left| \sum_{\psi} \frac{\langle i | \mathcal{H}_{\text{int}} | \psi \rangle \langle \psi | \mathcal{H}_{\text{int}} | i \rangle}{E_{\psi} - E_i} \right|^2 \delta(E_i - E_f)
\]

1. There are 2 channel which add coherently
2. The leads have macroscopic number of electrons. Therefore, with overwhelming probability the outgoing electron will come from a different state than the one which the incoming electron occupies → After the process an electron-hole excitation is left in the grain.

General process: tunneling through the chain of granules
Hopping conductivity in the regime of inelastic cotunneling

Hopping through a chain of grains via inelastic cotunneling

\[ P_{in} = \frac{1}{4\pi T} \frac{\bar{g}^N + 1}{\pi^{N+1}} \left[ \frac{4\pi T}{E} \right]^{2N} \left| \frac{\Gamma(N + \frac{i\Delta}{2\pi T})}{\Gamma(2N)} \right|^2 e^{-\frac{\Delta}{2T}} \]

\[ \Delta = \xi_N - \xi_0 \quad - \text{difference of the energies of initial and final states} \]
Low electric field (linear regime)

Optimization under constraint $N a \tilde{\kappa} \Delta / e^2 \sim 1, \ (N \gg 1)$, results in the ES law:

$$
\sigma \sim e^{- (T_0(T) / T)^{1/2}}
$$

$$
T_0(T) \sim e^2 / \tilde{\kappa} \xi_{in}(T)
$$

$$
\xi_{in}(T) = \frac{2 a}{\ln[ \bar{E}^2 / 16\pi T^2 \bar{g} ]}
$$

Crossover temperature between elastic and inelastic regimes

$$
\xi_{in} = \xi_{el} \quad \rightarrow \quad T = \sqrt{\delta E_c}
$$
Hopping conductivity via inelastic cotunneling

Regime of strong fields:

Low temperatures $T \to 0$:

$$P_{\text{in}}(T = 0) = \frac{2^{2N} \pi}{(2N-1)!} \frac{|\Delta|^{2N-1}}{E^{2N}} \left( \frac{\bar{g}}{\pi} \right)^{N+1}$$

Hopping distance can be found as in the case of elastic cotunneling:

$$e\mathcal{E}r \sim \Delta$$
$$r \Delta \bar{\kappa}/e^2 \sim 1$$

$\Rightarrow r \sim \sqrt{e/\bar{\kappa}\mathcal{E}}$

Using that $N \sim r/a$, $N \gg 1$

$$j \sim j_0 e^{-\left(\mathcal{E}_0/\mathcal{E}\right)^{1/2}},$$

$$\mathcal{E}_0(\mathcal{E}) \sim \frac{e}{\bar{\kappa}a^2} \ln^2 \left[ \frac{E^2}{e^2\mathcal{E}^2 a^2 \bar{g}} \right]$$

Applicability:

Nonlinear regime: $\mathcal{E}ea \gg T$

Inelastic cotunneling dominates elastic one: $\mathcal{E}ea \gg \sqrt{\delta E_c^0}$
Transmission electron micrographs showing the region between the in-plane electrodes for a) bilayers, b) trilayers, c) tetralayers and d) thick films. The darker regions on top and bottom of a-c are the electrodes. The insets on the right sides are diffraction patterns computed by fast fourier transform. The insets on the left sides of panels a & c are the zoomed-in images. The scale bars correspond to 200nm (a-c) and 40nm (d, all insets).
a) Zero-bias conductance $g_0$ versus inverse temperature $T^{-1/2}$ for representative multilayer and thick film data. The thick-film curve has been shifted upward for clarity by multiplying the data by 40. Inset: For the high-temperature range, where the multilayer data in the main panel deviate form the dotted lines, $g_0$ has been replotted as a function of $T^{-1}$, indicating Arrhenius behavior from 100-160K(b-e) Evolution of the I – V characteristics with temperature for bilayers (b,c) and thick films (d,e). Panels (c) and (e) are log-log plots of the data shown in the plots above them. The straight solid lines are guide to the eye, indicating power law behavior. The noise limit in these measurements was about 25fA. Insets to b&d: Temperature dependence of the hopping distance $N$ obtained from $g_0(T)$ and the I-V power-law exponents obtained from panels c&e in the range $2V < V < 7V$. 
At $T = 10K$, $N = 4$ for multilayers and 4-5 for the thick films.

at $T \approx 90 - 95K$  $N \sim 1$ for multilayers

for the thick films only above $T \approx 130K$.

$\text{ES} \rightarrow \text{Mott VRH at } T_{\text{cross}} = T_M \left( \frac{T_{ES}}{T_M} \right)^{\frac{d+1}{d-1}}$

$T_{\text{cross}} \approx 1400K$

$I_{in} \sim V \left[ \frac{g_T}{\hbar/e^2} \right]^j \left[ \frac{(eV)^2 + (k_BT)^2}{E_C^2} \right]^{j-1}$

$N=4$ or $j=3$ implies $I \sim V^5$. This is what experimentally observed
Summary of the results for metallic arrays

1. Periodic granular array:

   Activation conductivity \[ \sigma \sim e^{-\Delta_M(g)/T} \]

   T=0: Insulator to metal transition occurs at
   \[ g_c = \frac{1}{\pi \xi} \ln(E_c/g\delta) \]

2. Arrays with electrostatic disorder:

   Linear regime:

   \[ \sigma \sim e^{-\left(\frac{T_0}{T}\right)^{1/2}}, \quad T_0 \sim \frac{e^2}{\tilde{\kappa} \xi}, \quad \xi \sim \begin{cases} \frac{2 a}{\ln(\bar{E} \pi/\bar{g} \delta)}, & T < \sqrt{E_c^0 \delta} \quad \text{elastic} \\ \frac{2 a}{\ln[\bar{E}^2/16\pi T^2 \bar{g}]} & T > \sqrt{E_c^0 \delta} \quad \text{inelastic} \end{cases} \]

   Nonlinear regime:

   \[ j \sim j_0 e^{-\left(\mathcal{E}_0 / \mathcal{E}\right)^{1/2}}, \quad \mathcal{E}_0 \sim \begin{cases} \frac{e}{\tilde{\kappa} a^2} \ln^2[\bar{E} \pi / \delta \bar{g}], & \mathcal{E} e a < \sqrt{\delta E_c^0} \quad \text{elastic} \\ \frac{e}{\tilde{\kappa} a^2} \ln^2[\bar{E}^2 / e^2 \mathcal{E}^2 a^2 \bar{g}], & \mathcal{E} e a > \sqrt{\delta E_c^0} \quad \text{inelastic} \end{cases} \]
• Unresolved question: $\log T$ dependence of resistance at high temperatures
Hopping conductivity in granular superconductors

Weak coupling regime $g_T \ll 1$

**Simplest Model:**

Coulomb energy + Josephson couplings

$\phi$ – Cooper pair phase

$\hat{n} = -i \frac{\partial}{\partial \phi}$ - Cooper pair number operator

\[
H = 4 \sum_{ij} \hat{n}_i E_{ij} \hat{n}_j + \sum_{ij} J_{ij} \cos(\phi_i - \phi_j)
\]

Anderson-Abeles criterion for the global superconductivity development $J > E_c$

Applicable at $g \ll 1$ - as long as the charge renormalization effects may be neglected.

We assume $g \ll 1$, $g \Delta \ll E_c$ → No global coherence

In the presence of electrostatic disorder the transport can be mediated by Cooper pair hopping

**Questions:**

1. **Electron** hopping transport in the presence of the superconducting gap - ?
2. Multiple cotunneling in the presence of the gap - ?
3. Conductivity temperature dependence - ?
Density of states: Single grain model

Energy of a single superconducting grain.

\[ E = n^2 E_c - Vn + P(n + p) \Delta \]

Charging energy \quad Random potential \quad Parity term.

- \( n \) - number of excessive electrons, counted with the respect to \( N_0 \) – the total charge of the neutral state

Total number of electrons \( N = N_0 + n \).

Parity effect: A state with odd number of electrons has an extra energy \( \Delta \).

Parity function:

\[ P(n) = \begin{cases} 0 & \text{even } n \\ 1 & \text{odd } n \end{cases} \]

- \( p = 1,2 \) – the total charge \( N \) of the neutral state can be even or odd.

The energy \( E \) is at minimum with respect to electron number \( n \) for a given potential \( V \).

Excitation energies to add or remove an electron are all positive for any \( V \)!

Same for the pair creation and annihilation processes.
**Single grain model: Electron occupation number**

<table>
<thead>
<tr>
<th>Electron excitation energy:</th>
<th>Pair excitation energy:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{E}_{\pm} \equiv E(n + 1) \pm E(n) )</td>
<td>( \mathcal{E}_{2\pm} \equiv E(n + 2) - E(n) )</td>
</tr>
<tr>
<td>( = (\pm 2n + 1)E_c \mp V + \Delta \cos \pi(n + p) )</td>
<td>( = 4(\pm n + 1)E_c \mp 2V )</td>
</tr>
</tbody>
</table>

Occupation number \( n \) jumps
\( n \rightarrow n \pm 1 \) at \( \mathcal{E}_{\pm} = 0 \)

Occupation number \( n \) jumps
\( n \rightarrow n \pm 2 \) at \( \mathcal{E}_{2\pm} = 0 \)

Depending on the mutual relation of \( E_c \) and \( \Delta \) one finds qualitatively different dependences \( n(V) \):

1. **Charging energy dominates** \( E_c > \Delta \).
   - Modified Coulomb staircase:
     - Occupation number changes by one
     - \( n \rightarrow n + 1 \) at
     \[ V_n = (2n + 1)E_c + \Delta \cos \pi(n + p) \]
     - Pair excitations are gapped for all \( V \)!
     \[ \mathcal{E}_{2\pm} > 2(E_c - \Delta) \]

2. **Parity term dominates** \( \Delta > E_c \).
   - Usual staircase but for Cooper pairs:
     - Occupation number changes by two
     - \( n \rightarrow n + 2 \) at
     \[ V_n = E_c(4n + 2) \]
     - Electron excitations are gapped for all \( V \)!
     \[ \mathcal{E}_{\pm} > \Delta - E_c \]
Electron number and excitation energies as functions of $V$, $E_c > \Delta$

Electron occupation number: $n(V)$

- $n=0$
- $n=1$
- $n=-1$
- $Ec$
- $2Ec$
- $3Ec$
- $-Ec$
- $-2Ec$
- $-3Ec$

Single- and two-particle excitation energies

- $3Ec - \Delta$
- $E_c - \Delta$
- $E_c + \Delta$
- $3Ec + \Delta$

$n=-1$ $n=0$ $n=1$

$2Ec - 2\Delta$

solid lines – creation
dashed lines - annihilation
Electron number and excitation energies as functions of $V$, $\Delta > E_c$

Electron occupation number: $n(V)$

- Solid lines – creation
- Dashed lines – annihilation

Single- and two particle excitation energies

$\Delta - E_c$

solid lines – creation

dashed lines – annihilation
Density of states

Long range Coulomb interaction

\[ H_c = \sum_{ij} n_i E_{ij}^c n_j - V_i n_i + \Delta P(n_i + p_i) \quad E_c \sim \frac{e^2}{2\tilde{\kappa} r} \quad r \to \infty \]

Main conclusions of the single grain model stay the same!

1. \( E_c > \Delta \): Gapless electrons.
   Pair gap: \( 2(E_c - \Delta) \)

2. \( E_c < \Delta \): Gapless pairs.
   Electron gap: \( \Delta - E_c \)

DOS: Efros-Shklovskii approach:

Energy to replace electron form i to j

\[ \mathcal{E}_{-+}^{ij} = \mathcal{E}_-^i + \mathcal{E}_+^j - 2E_c^{ij} > 0 \]

\[ \nu_1(\varepsilon) = \alpha_{d1} \left( \frac{\tilde{\kappa}}{e^2} \right)^d \varepsilon^{d-1} \]

Energy to replace a pair from i to j

\[ \mathcal{E}_{2-2+}^{ij} = \mathcal{E}_{2-}^i + \mathcal{E}_{2+}^j - 8E_c^{ij} > 0 \]

\[ \nu_2(\varepsilon) = \alpha_{d2} \left( \frac{\tilde{\kappa}}{(2e)^2} \right)^d \varepsilon^{d-1} \]

The difference is due to the Cooper pair doubled charge only, therefore \( \alpha_{d1} \approx \alpha_{d2} \).
Multiple cotunneling in granular superconductors. Hopping Conductivity

Electron hopping regime: $E_c > \Delta$. Limiting case $E_c \gg \Delta$.

Hopping conductivity

$$\sigma \sim e^{-(T_0/T)^{1/2}} \quad T_0 \sim e^2/\kappa \xi_{el}$$

The presence of the gap results in a small correction to the localization length:

$$\xi_{el} = \frac{2 a}{\ln(\bar{E}(\Delta) \pi / g \delta)}$$

$$\bar{E}(\Delta) = \bar{E}(0) + c\Delta$$

Noticeable negative magneto-resistance.

1. $T \gg \Delta$: ES law with essentially unaffected localization length:

$$\xi_{in}(T) = \frac{2 a}{\ln[\bar{E}^2/16\pi T^2 g]}$$

2. $T \ll \Delta$: Strong suppression of the inelastic cotunneling!

$$\sigma \sim \exp \left[-N \left(\ln(\bar{E}^2/4gT\Delta) + 2\Delta/T\right)\right]$$

N is the typical tunneling order:

$$N = \sqrt{b e^2/16 a \kappa \Delta} \sim \sqrt{E_c/\Delta}$$

Giant negative magneto-resistance!
The gap $\Delta$ can be tuned by the magnetic field

$T_1 \approx 0.1 \sqrt{E_c \delta}$ - Crossover between elastic and inelastic regimes at $\Delta=0$

$T_2 \approx \xi_{el} \Delta/a$ - Crossover between the elastic and inelastic activation behavior at $H=0$

$T_3 \approx \xi_{in} \Delta/a$ - Crossover between ES and activation inelastic regimes
Cooper pair hopping (CPH) regime

Hopping of Cooper pairs can be described in terms of the effective Hamiltonian

\[ H = 4 \sum_{i,j} \hat{n}_i E_c^{ij} \hat{n}_j - 2 \sum_i \hat{n}_i V_i + \frac{1}{2} \sum_{<ij>} J_{ij} e^{i\varphi_i - i\varphi_j} \]

- \( \varphi \) - Cooper pair phase
- \( \hat{n} = -i\partial/\partial \varphi \) - Cooper pair number operator
- \( J_{ij} = g_{ij} \pi \Delta / 2 \) - Josephson couplings

Tunneling amplitude via perturbation theory in J:

\[ A \sim \prod_{i=1}^{N} J_{i,i+1}/\tilde{\mathcal{E}}_2^i, \quad \tilde{\mathcal{E}}_2^i = 2/[1/\mathcal{E}_{2+}^i + 1/\mathcal{E}_{2-}^i] \]

- \( \mathcal{E}_{2+}^i, \mathcal{E}_{2-}^i \) - Cooper pair creation and annihilation energies

Tunneling probability

\[ P = A^* A \sim e^{-2r/\xi_{CPH}} \]

Conductivity:

\[ \sigma \sim e^{-(T_0/T)^{1/2}} \]

\[ T_0 \sim e^2/\kappa \xi_{CPH} \]

\[ \xi_{CPH} = \frac{a}{\ln(8\bar{E}/\pi g \Delta)} \]

Positive magnetoresistance!
Experimental data

Granular aluminum samples.


Weak coupling insulating regime.

Grain size ~ 120Å

$E_c \gg \Delta \rightarrow$ Electron hopping!

Theory predicts the giant negative magneto-resistance at $T < \Delta$.

Explanation: suppression of the inelastic cotunneling by the superconducting gap.

FIG. 1. Resistance of sample 1 measured at zero (triangles) and 100 kOe field (circles) as a function of the inverse square root of the temperature. Sample 1 room temperature resistance is $2 \times 10^3 \ \Omega$. 
Hopping conductivity in superconductors: Results

Hopping law

Ec > Δ. Electron hopping.

Ec < Δ. Cooper pair hopping.

1. ES law for Cooper pair transport.

2. Positive magnetoresistance.

3. Possible scenario at g ~ 1:
Renormalization of the charging energy due to tunneling coupling.

\[
T_3 \approx \xi_{in} \Delta / a
\]
ES law. Inelastic regime.

\[
T_2 \approx \xi_{el} \Delta / a
\]
Activation law. Inelastic regime.

\[
T_1 \approx 0.1 \sqrt{E_c \delta}
\]
ES law; elastic regime.

Negative magnetoresistance
1. Granular metal:

\[ \int d\mathbf{p} \ G(\mathbf{p}, \omega) \ G(\mathbf{p}, \omega) = \int \frac{d\omega}{(\xi - \omega)^2} = 0 \]

2. Disordered metal:

\[ \int d\mathbf{p} \ G(\mathbf{p}, \omega) \ G(\mathbf{p}, \omega) \ G(\mathbf{p}, \omega + \omega) \neq 0 \]
Homogeneous wires

low electron density: $k_F \sim L \sim 1$ -100 nm
lateral confinement to lowest quantum state

strong Coulomb interactions:
- $\rightarrow$ 1D Wigner crystal  $R = \frac{h}{e^2}$

weak Coulomb interactions,
- Luttinger liquid  $R = \frac{h}{2e^2} \sim 13k\Omega$

charge conductor, spin insulator
spin order in Wigner crystal,
two channel conductance

Matveev PRL 92, 106801, 2004
Quantum wires: importance of morphology

**homogeneous**

<table>
<thead>
<tr>
<th>Diffusive electron motion</th>
</tr>
</thead>
</table>

**granular**

Energy scales:

\[ E_F \] - Fermi energy

\[ \tau \] - mean free time

\[ \delta \] - mean energy level spacing

\[ E_c \approx e^2 / R \] - Coulomb energy

\[ g_T \] - tunneling conductance

\[ g_0 \] - grain conductance

Granular physics more rich! \[ \implies \] Homogeneous wires: \[ g_T \approx g_0 \]
Granular wires

Electron diffusion
Tunneling junction
Coulomb interaction
Electron tunneling

charge leakage
tunneling time $\sim RC$
lifetime broadening $\Delta E_C \sim \frac{h}{\Delta t} \sim \frac{h}{RC}$

Coulomb blockade
charging energy $E_C \sim \frac{e^2}{C}$

superconductor
$R < \frac{h}{(2e)^2}$

metal
$E_C < \Delta E_C$
$R < \frac{h}{e^2}$

insulator
$\Delta E_C < E_C$
$\frac{h}{e^2} < R$

experiment
$10^6 - 10^{10} \ \Omega$

$tunneling \ resistance \ controls \ collective \ response$
Tunneling density of states at $g_T \gg 1$

3D granular metals:

$$\frac{\nu(T)}{\nu_0} = \left[1 - \frac{1}{6\pi g_T} \ln \frac{g_T E_C}{T}\right]^{3A}$$

Granular films:

$$\frac{\nu(T)}{\nu_0} = \left[\frac{g_T E_C}{T}\right]^{1/\pi} \left[1 - \frac{1}{4\pi g_T} \ln \frac{g_T E_C}{T}\right]^{4g_T}$$

Generalization to finite energies: $T \rightarrow \max\{T, \epsilon\}$

The Mott gap

$$\Delta \sim E_C g_T e^{-2\pi d g_T}$$

opens at $T=0$ for samples with tunneling conductance $g_T < g_T^c = \frac{1}{6\pi} \ln \left(\frac{E_C}{\delta}\right)$

Expected activation conductivity behavior:

$$\sigma \sim e^{-\Delta/T}$$
Hikami-parametrization for $Q$–matrix

\[ Q = \begin{pmatrix} \sqrt{1 - BB^\dagger} & B \\ B^\dagger & -\sqrt{1 - B^\dagger B} \end{pmatrix} \]

For energies $\varepsilon \ll E_T$, the $Q$–matrices are coordinate independent within each grain.

$B_{\omega_n, \omega_n}$ has non-zero elements only for frequencies $\omega_n > 0, \omega_n < 0$

Expansion of the $Q$–matrix in powers of the field $B$ provides a systematic way to obtain $1/g_T$ corrections.

We divide the field $B$ into the slow and fast parts $B = Bs + Bf$ and define the fast part of the field $B_{\omega_n, \omega_n}$ in a way that it exists either for

$\Lambda < \omega_n < \Lambda + d\Lambda$ or $-\Lambda - d\Lambda < \omega_n < -\Lambda$
Tunneling density of states

Since at low temperatures the flow of coupling constants of the $\sigma$-model of granular metals is determined by the same renormalization group equations as in the case of homogeneously disordered metals, one arrives at the important conclusion that the tunneling density of states has a multiplicative structure:

$$\nu / \nu_0 = \nu_h \nu_l$$

where $\nu_0$ is the density of states of non-interacting electrons, $\nu_h$ is the contribution to the density of states that comes from high energies, $\varepsilon > g T \delta$, while $\nu_l$ is the contribution from low energies, $\varepsilon < g T \delta$, which up to the proper renormalization of all constants coincides with the corresponding result for the density of states of disordered homogeneous metals.
Films:

\[
\nu_l = \exp\left[ -\frac{1}{16g_T\pi^2} \ln \frac{g_T\delta}{T} \ln \frac{g_T E_C^4}{T\delta^3} \right]
\]

\[
\nu_h = \left[ \frac{E_C}{\delta} \right]^{1/\pi} \left[ 1 - \frac{\ln(E_C/\delta)}{4\pi g_T^{(0)}} \right]^{4g_T^{(0)}}
\]