Particle-based simulation of the Gel'fand-Pinsker channel capacity and the Wyner-Ziv rate-distortion function

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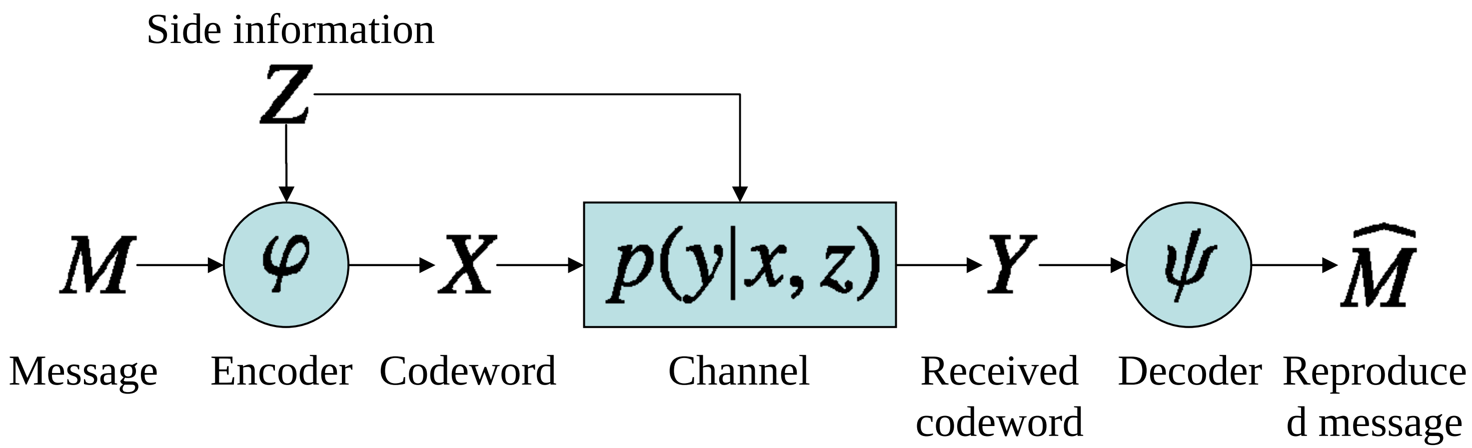
Photo by Junko Kimura (in Seven Sisters, East Sussex, United Kingdom)

Abstract

- A new numerical algorithm for simulating
  - Gel'fand-Pinsker channel capacity for memoryless channels
  - Wyner-Ziv rate-distortion function for memoryless sources (skipped)



- Basic idea
  - Represent a density by a finite number of "particles" (= sample values + the associated weights)

Outline

1. Introduction
2. Previous method
3. Proposed method
4. Summary

Introduction

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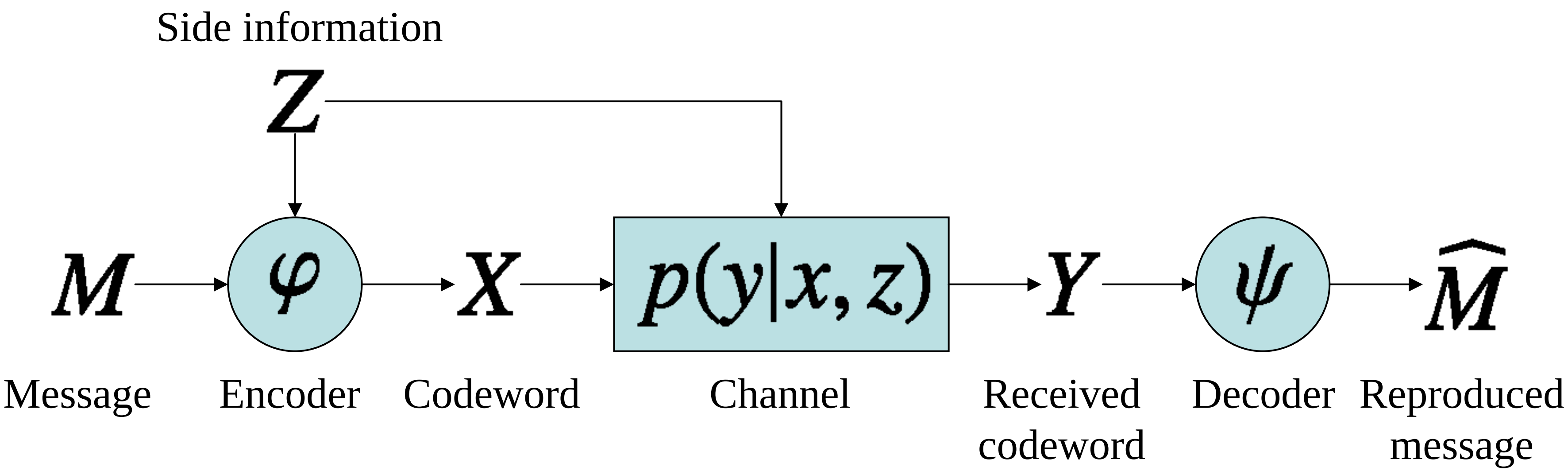
Background

- Gel'fand and Pinsker (1980)
  - Investigated a channel with non-causal encoder side information



Capacity:

$$C_{GP}(X;Y|Z) = \max_{P_{U|Z}, P_{X|UZ}} \{I(U;Y) - I(U;Z)\}$$

Background

- Dupuis, Fu and Willems (2004)
  - Modified Arimoto-Blahut algorithm for computing Gel'fand-Pinsker channels
  - Applicable only for finite alphabets

  - GP capacity for memoryless channels with infinite alphabets

$$C_{GP}(X;Y|Z) = \sup_{P_{U|Z},P_{X|UZ}} \{I(U;Y) - I(U;Z)\}$$

$$= \sup_{P_{U|Z},P_{X|UZ}} \int_{(u,x,y,z)} P_{UXYZ}(u,x,y,z) \log \frac{P_{U|Y}(u|y)}{P_{U|Z}(u|z)} dudxdydz$$

→ Integration calculations are often intractable.

Approach

- Represent a density by a finite number of "particles" (= samples + associated weights)

$$P_X(x) \approx \widetilde{P}_X(x) = \sum_{i=1}^{N} w_i \cdot \delta(x - \widetilde{x}_i)$$

$\widetilde{x}_i$: sample
w_i: weight
$\delta(\cdot)$: delta function

(Note) Particle filter

One of the most popular Bayesian algorithms for estimating hidden states based only on observations.

The procedure:
1. Update weights
2. Re-sampling
3. Update samples

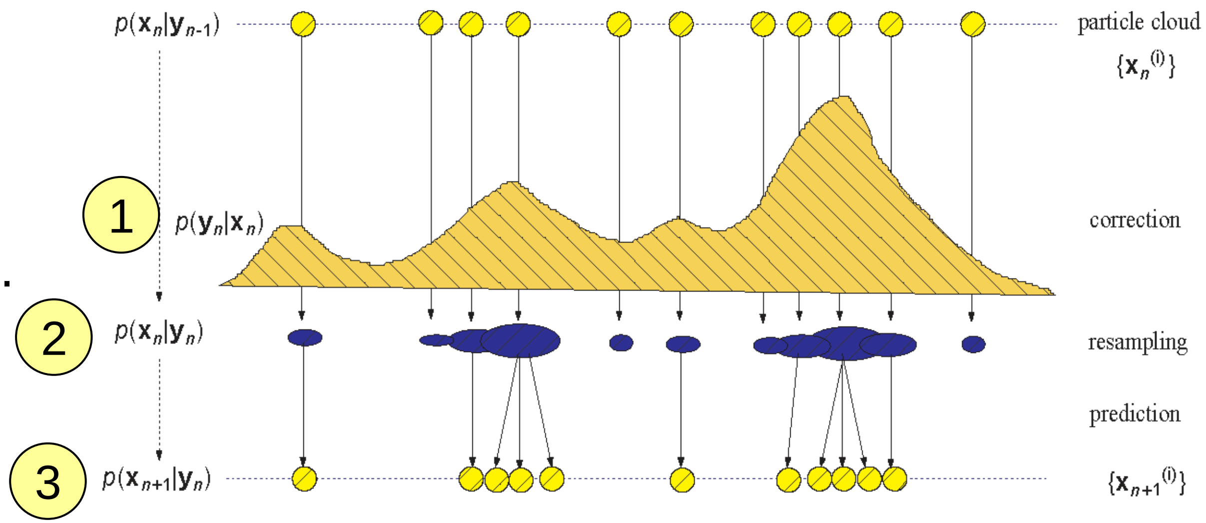


Fig. 9. An illustration of generic particle filter with importance sampling and resampling.

Previous method

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Another form of GP capacity

- Original form of Gel'fand-Pinsker (GP) capacity

$$C_{GP}(X;Y|Z) = \sup_{P_{U|Z}, P_{X|UZ}} \int_{(u,x,y,z)} P_{UXYZ}(u,x,y,z) \log \frac{P_{U|Y}(u|y)}{P_{U|Z}(u|z)} du dx dy dz$$

  – Simultaneous optimization may be intractable.

- Dupuis, Fu and Willems (2004)
  – The capacity is given with a function $t : \mathcal{Z} \to \mathcal{X}$ by

$$\begin{aligned} C_{GP}(X;Y|Z) &= \sup_{P_{T|Z}} \{ I(T;Y) - I(T;Z) \} \\ &= \sup_{P_{T|Z}} \int_{(u,y,z)} P_{TYZ}(t,y,z) \log \frac{P_{T|Y}(t|y)}{P_{T|Z}(t|z)} dt dy dz \\ P_{TYZ} &= P_Z(z) P_{T|Z}(t|z) P_{Y|XZ}(y|t(z),z) \end{aligned}$$

Arimoto-Blahut for GP channels

- The GP capacity is rewritten as

$$C_{GP}(X;Y|Z) = \sup_{Q_{T|Z}, Q_{T|Y}} \int_{(t,y,z)} P_Z(z) Q_{T|Z}(t|z) P_{Y|XZ}(y|t(z), z) \log \frac{Q_{T|Y}(t|y)}{Q_{T|Z}(t|z)} dt dy dz$$

- $Q_{T|Z}$ and $Q_{T|Y}$ can be updated in each iteration as

$$Q^{(m)}_{T|Y}(t|y) \propto \int_z P_Z(z) Q^{(m-1)}_{T|Z}(t|z) P_{Y|XZ}(y|t(z), z) dz,$$

$$Q^{(m)}_{T|Z}(t|z) \propto \exp\left\{ \int_y P_{Y|XZ}(y|t(z), z) \log Q^{(m)}_{T|Y}(t|y) dy. \right\}$$

  - Integration calculations are often intractable.
  - Calculating $Q_{T|Z}$ and $Q_{T|Y}$ for all t is impossible

Proposed method

1. Introduction
2. Previous method
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Framework (1 / 4)

- $P_{T|Z}$ is represented by a finite number of particles

$$P_{T|Z}(t|z) \approx \widetilde{P}_{T|Z}(t|\widetilde{z}_i) = \sum_{j=1}^{N} w_{i,j} \cdot \delta(t - \widetilde{t}_{i,j}),$$

$$\mathcal{S}_{T|Z} = \left\{\mathcal{S}_{T|Z}(\widetilde{z}_i)\right\}_{i=1}^{M_Z}, \quad \mathcal{S}_{T|Z}(\widetilde{z}_i) = \left\{\widetilde{t}_{i,j}\right\}_{j=1}^{N}, \quad \text{: samples}$$

$$\mathcal{W}_{T|Z} = \left\{\mathcal{W}_{T|Z}(\widetilde{z}_i)\right\}_{i=1}^{M_Z}, \quad \mathcal{W}_{T|Z}(\widetilde{z}_i) = \left\{w_{i,j}\right\}_{j=1}^{N} \text{: weights}$$

  – $\mathcal{W}_{T|Z}$ and $\mathcal{S}_{T|Z}$ determine the density $P_{T|Z}$

Framework (2 / 4)

- The original optimization problem

$$C_{GP}(X;Y|Z) = \sup_{P_{T|Z}} \int_{(u,y,z)} P_{TYZ}(t,y,z) \log \frac{P_{T|Y}(t|y)}{P_{T|Z}(t|z)} dt dy dz$$

can be replaced as

$$\underline{C}_{GP}(X;Y|Z) = \max_{\mathcal{S}_{T|Z}, \mathcal{W}_{T|Z}} I(\mathcal{S}_{T|Z}, \mathcal{W}_{T|Z})$$

$$I(\mathcal{S}_{T|Z}, \mathcal{W}_{T|Z})$$

$$= \int_y \sum_{i=1}^{M_Z} \sum_{j=1}^{N} \widetilde{P}_{TYZ}(\widetilde{t}_{i,j}, y, \widetilde{z}_i; w_{i,j}) \log \frac{\widetilde{P}_{T|Y}(\widetilde{t}_{i,j}|y)}{w_{i,j}} dy$$

$$\widetilde{P}_{TYZ}(\widetilde{t}_{i,j}, y, \widetilde{z}_i; w_{i,j}) = w_{i,j}\, P_Z(\widetilde{z}_i)\, P_{Y|XZ}(y|\widetilde{t}_{i,j}(\widetilde{z}_i), \widetilde{z}_i)$$

Framework (3 / 4)

- The maximization can be solved by the following alternative maximization:

$$\mathcal{W}_{T|Z}^{(n)} = \arg\max_{\mathcal{W}_{T|Z}} I(\mathcal{S}_{T|Z}^{(n-1)}, \mathcal{W}_{T|Z}),$$
$$\mathcal{S}_{T|Z}^{(n)} = \arg\max_{\mathcal{S}_{T|Z}} I(\mathcal{S}_{T|Z}, \mathcal{W}_{T|Z}^{(n)})$$

- The first maximization:
  almost the same as the previous method

$$Q_{T|Y}^{(n,m)}(\tilde{t}_{i,j}^{(n-1)}|y) \propto \sum_{i=i}^{M_Z} w_{i,j}^{(n,m-1)} P_Z(\tilde{z}_i) P_{Y|XZ}(y|\tilde{t}_{i,j}^{(n-1)}(\tilde{z}_i), \tilde{z}_i),$$
$$w_{i,j}^{(n,m)} \propto \exp\left\{ \int_y P_{Y|XZ}(y|\tilde{t}_{i,j}^{(n)}(\tilde{z}_i), \tilde{z}_i) \log Q_{T|Y}^{(n,m)}(\tilde{t}_{i,j}^{(n-1)}|y) dy \right\}.$$

Framework (4 / 4)

- The second maximization:
  difficult to maximize it directly
  → update samples to increase the capacity
  1. Re-sampling to avoid "degeneracy".
     New samples are drawn from

$$\widetilde{P}_{T|Z}(t|\widetilde{z}_i) = \sum_{j=1}^{N} w_{i,j} \cdot \delta(t - \widetilde{t}_{i,j})$$

  2. Update samples based on steepest descent method.

$$\widetilde{t}_{i,j}^{(n)} = \widetilde{t}_{i,j}^{(n-1)} + \left(\lambda_{i,j}^{(n)} + \eta_{i,j}^{(n)}\right) \frac{\partial}{\partial \widetilde{t}_{i,j}} I(\mathcal{S}_{T|Z}, \mathcal{W}_{T|Z}^{(n)}) \Big|_{\mathcal{S}_{T|Z} = \mathcal{S}_{T|Z}^{(n-1)}}$$

$\lambda_{i,j}^{(n)}$: step size, $\eta_{i,j}^{(n)}$: r.v. which depends on $\lambda_{i,j}^{(n)}$

Procedure: summary

1. Initialization
   i. Draw samples $\{\tilde{z}_i\}_i$
   ii. Initialize particles $\mathcal{S}^{(0)}_{T|Z}, \mathcal{W}^{(0)}_{T|Z}$
2. Update weights $\mathcal{W}^{(n)}_{T|Z}$ given samples $\mathcal{S}^{(n-1)}_{T|Z}$
   i. Update a density $Q^{(n,m)}_{T|Y}$
   ii. Update weights $\mathcal{W}^{(n,m)}_{T|Z}$
   iii. Repeat until i.-ii. convergence
3. Update samples $\mathcal{S}^{(n)}_{T|Z}$ given weights $\mathcal{W}^{(n)}_{T|Z}$
   i. (if needed) Re-sampling
   ii. Update samples $\mathcal{S}^{(n)}_{T|Z}$
4. Repeat 2.-3. until convergence

Propositions

Lemma . $\underline{C}_{GP}(X;Y|Z) \leq C_{GP}(X;Y|Z)$

Lemma . *With the proposed method (without the resampling step)*
$I(\mathcal{S}_{T|Z}^{(n-1)}, \mathcal{W}_{T|Z}^{(n-1)}) \leq I(\mathcal{S}_{T|Z}^{(n-1)}, \mathcal{W}_{T|Z}^{(n)}) \leq I(\mathcal{S}_{T|Z}^{(n)}, \mathcal{W}_{T|Z}^{(n)})$

Theorem . *With the proposed method (without resampling), $\underline{C}_{GP}(X;Y|Z)$ converges to a local extrema if the step sizes $\{\lambda_i\}_{i=1,2,\ldots}$ satisfy the Wolfe conditions.*

Proposition . *Especially, if all the alphabets are equal to $\mathcal{R}^d$ (d: dimension) and the density P_{TYZ} is continuous, then with the proposed method we have*

$$\underline{C}_{GP}(X;Y|Z) \quad \xrightarrow{p} \quad C_{GP}(X;Y|Z) \quad (M_Z, N \to \infty)$$

How do we check the continuity of P_{TYZ} ??

Problems

1. Complexity and memory

   [Complexity priority]

   complexity: $O(M_Z N M_Y)$, memory: $O(M_Z N M_Y)$

   [Memory priority]

   complexity: $O(M_Z^2 N M_Y)$, memory: $O(\max\{M_Z N, N M_Y\})$

2. General convergence
   - Not proved yet.
   - When estimating the rate-distortion function by empirical estimators, in certain cases it may fail to converge to the rate-distortion function. [Harrison and Kontoyiannis 2008]

Summary

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Concluding remarks

- Present a new numerical method for simulating the Gel'fand-Pinsker channel capacity
  - By representing a density with particles
  - It can be applied to simulate the Wyner-Ziv rate-distortion function

- Future work
  - Prove the theoretical convergence
  - Reduce the computational complexity

Thank you for your kind attention.

Appendix

Note

- Sampling itself is often intractable.
  → Markov Chain Monte-Carlo (MCMC)

- The algorithm sometimes encounters underflow problems, especially when calculating

$$w_{i,j}^{(n,m)} \propto \exp\left\{ \int_y P_{Y|XZ}(y|\tilde{t}_{i,j}^{(n)}(\tilde{z}_i), \tilde{z}_i) \log Q_{T|Y}^{(n,m)}(\tilde{t}_{i,j}^{(n-1)}|y) dy \right\}.$$

- The step size should be controlled so as to satisfy the Wolfe conditions.
  - Criteria for the convergence of the steepest descent