Weak Variable-Length Slepian-Wolf Coding with Linked Encoders for Mixed Sources

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Abstract — **Slepian and Wolf first considered the data compression of correlated sources called the SW system, where two sequences emitted from correlated sources are separately encoded to codewords, and sent to a single decoder which has to output original sequence pairs. Recently, Oohama has extended the SW system and investigated a more general case where there are some mutual linkages between two encoders of the SW system. In this paper, we investigate variable-length coding which allows asymptotically vanishing probability of error for the system considered by Oohama. We clarify the admissible rate region for mixed sources characterized by two ergodic sources, and show that this region is strictly wider than that for fixed-length codes.**

I. Introduction

Coding problems for correlated information sources were first investigated by Slepian and Wolf [1]. They considered the data compression system, where two sequences of length n emitted from correlated sources are separately encoded to nR_1 and nR_2 bit codewords, and sent to a single decoder which has to output original sequence pairs with small probability of error. Slepian-Wolf established the admissible rate region (called *the SW region*), namely the closure of the set which consists of the rate (R_1, R_2) such that the error probability of decoding can be made arbitrarily small by letting n to be large. Their coding theorem may be regarded as a substantial starting point of multiterminal information theory, and many variations of their data compression system have been investigated. Recently, Miyake and Kanaya [2] extended the coding theorem to the class of non-ergodic or non-stationary sources called *general sources* by using the method developed by Han and Verdú [3, 4].

In the system of Slepian and Wolf (called *the SW system*) neither of the encoders can observe the codeword generated by the other encoder. Oohama [5] has investigated a general case where there are some mutual linkages between two encoders of the SW system. However, Oohama only dealt with fixed-length source encoders, and the variable-length coding problem for the SW system still remains open despite its significance. On the other hand, in the coding problem for single general sources, Han [6] has shown that weak variable-length code, i.e. variable-length code having asymptotically vanishing probability of error, may achieve lower coding rate than fixed-length code. Hence, we can expect a similar result for correlated sources. This motivates us this research.

In this paper, we investigate weak variable-length coding problems for correlated two sources in the system considered by Oohama. We clarify the admissible rate region for mixed sources characterized by two ergodic sources, and show that this region is strictly wider than that for fixed-length codes.

II. Coding Systems for Correlated Sources

A. Basic Definitions

Let $\mathcal{X}$ and $\mathcal{Y}$ be finite sets and $\mathcal{B}$ be a binary set. Without loss of generality, we assume that $\mathcal{X} = \mathcal{Y} = \{1, 2, \cdots, M\}$ and $\mathcal{B} = \{0, 1\}$. We denote a set of all sequences of finite length by $\mathcal{B}^*$. Let $(\boldsymbol{X}, \boldsymbol{Y}) = \{(X_j, Y_j)\}_{j=1}^{\infty}$ be a stationary-ergodic process of random variables (X_j, Y_j) $(j = 1, 2, \cdots)$ which takes values in $\mathcal{X} \times \mathcal{Y}$. Then, both $\boldsymbol{X} = \{X_j\}_{j=1}^{\infty}$ and $\boldsymbol{Y} = \{Y_j\}_{j=1}^{\infty}$ are stationary-ergodic processes. We shall call $\boldsymbol{X}$ and $\boldsymbol{Y}$ *ergodic sources*, and $(\boldsymbol{X}, \boldsymbol{Y})$ *correlated ergodic source*. We define a *correlated mixed source* $(\boldsymbol{X}, \boldsymbol{Y})$ by the following distribution:

$$P_n(\boldsymbol{x}, \boldsymbol{y}) \stackrel{\triangle}{=} \alpha P_n^{(1)}(\boldsymbol{x}, \boldsymbol{y}) + (1 - \alpha) P_n^{(2)}(\boldsymbol{x}, \boldsymbol{y})$$

for all $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$, where $0 < \alpha < 1$ and $P_n^{(i)}$ $(i = 1, 2)$ are the distributions of the jointly ergodic process $\left(X_{(i)}^n, Y_{(i)}^n\right) = \left\{\left(X_{(i)j}, Y_{(i)j}\right)\right\}_{j=1}^{n}$. Further, we introduce the notation

$$(\boldsymbol{X}_{(i)}, \boldsymbol{Y}_{(i)}) \stackrel{\triangle}{=} \left\{\left(X_{(i)j}, Y_{(i)j}\right)\right\}_{j=1}^{\infty} \quad (i = 1, 2).$$

For an ergodic source $\boldsymbol{X}$, let $H(\boldsymbol{X})$ denote its *entropy rate*. Similarly, for a correlated ergodic source $(\boldsymbol{X}, \boldsymbol{Y})$, we denote a *joint entropy rate* and a *conditional entropy rate* by $H(\boldsymbol{X}, \boldsymbol{Y})$ and $H(\boldsymbol{X}|\boldsymbol{Y})$, respectively. In what follows, all logarithms and exponentials are to the base two.

B. Slepian-Wolf Coding System

Slepian and Wolf [1] studied the coding problem for two correlated sources. We call their data compression system *the Slepian-Wolf system (the SW system)*.

Definition 1: A sequence $\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \varphi_n^{-1})\}_{n=1}^{\infty}$ of triples is called a *(fixed-length) SW code (SW code)*, if the encoders $\varphi_n^{(1)} : \mathcal{X}^n \to \mathcal{M}_n^{(1)}$, $\varphi_n^{(2)} : \mathcal{Y}^n \to \mathcal{M}_n^{(2)}$, and the decoder $\varphi_n^{-1} : \mathcal{M}_n^{(1)} \times \mathcal{M}_n^{(2)} \to \mathcal{X}^n \times \mathcal{Y}^n$ satisfy

$$\lim_{n\to\infty} \Pr\{\varphi_n^{-1}(\varphi_n^{(1)}(X^n), \varphi_n^{(2)}(Y^n)) \neq (X^n, Y^n)\} = 0,$$

where $\mathcal{M}_n^{(i)} = \{1, 2, \cdots, M_n^{(i)}\}$ for $i = 1, 2$.

Definition 2: A rate pair (R_1, R_2) is *admissible for the SW system*, if there exists a SW code which satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \log M_n^{(1)} \leq R_1 \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n} \log M_n^{(2)} \leq R_2.$$

Definition 3 (The SW region): The SW region $\mathcal{R}_{SW}(\boldsymbol{X}, \boldsymbol{Y})$ is defined as a set of (R_1, R_2) which is admissible for the SW system.

Miyake and Kanaya [2] investigated the SW system for two correlated *general sources*[4] such that the joint distribution $P_n(\boldsymbol{x}, \boldsymbol{y})$ is given for each n independently, and clarified the SW region as follows:

Theorem 1 [2]: For any correlated general source $(\boldsymbol{X}, \boldsymbol{Y})$,

$$\mathcal{R}_{SW}(\boldsymbol{X}, \boldsymbol{Y}) = \{(R_1, R_2) : R_1 \geq \overline{H}(\boldsymbol{X}|\boldsymbol{Y}), \\ R_2 \geq \overline{H}(\boldsymbol{Y}|\boldsymbol{X}),\ R_1 + R_2 \geq \overline{H}(\boldsymbol{X}, \boldsymbol{Y})\},$$

where $\overline{H}(\boldsymbol{X}, \boldsymbol{Y})$ is the *joint sup-entropy rate* [4], while $\overline{H}(\boldsymbol{X}|\boldsymbol{Y})$ and $\overline{H}(\boldsymbol{Y}|\boldsymbol{X})$ are the *conditional sup-entropy rate* [4].

Corollary 1: If $(\boldsymbol{X}, \boldsymbol{Y})$ is a correlated ergodic source,

$$\mathcal{R}_{SW}(\boldsymbol{X}, \boldsymbol{Y}) = \{(R_1, R_2) : R_1 \geq H(\boldsymbol{X}|\boldsymbol{Y}), \\ R_2 \geq H(\boldsymbol{Y}|\boldsymbol{X}),\ R_1 + R_2 \geq H(\boldsymbol{X}, \boldsymbol{Y})\}.$$

Further, if $(\boldsymbol{X}, \boldsymbol{Y})$ is a correlated mixed source,

$$\begin{aligned} \mathcal{R}_{SW}(\boldsymbol{X}, \boldsymbol{Y}) = \big\{&(R_1, R_2) : \\ &R_1 \geq \max\left(H(\boldsymbol{X}_{(1)}|\boldsymbol{Y}_{(1)}), H(\boldsymbol{X}_{(2)}|\boldsymbol{Y}_{(2)})\right), \\ &R_2 \geq \max\left(H(\boldsymbol{Y}_{(1)}|\boldsymbol{X}_{(1)}), H(\boldsymbol{Y}_{(2)}|\boldsymbol{X}_{(2)})\right), \\ &R_1 + R_2 \geq \max\left(H(\boldsymbol{X}_{(1)}, \boldsymbol{Y}_{(1)}), H(\boldsymbol{X}_{(2)}, \boldsymbol{Y}_{(2)})\right) \big\}. \end{aligned}$$

C. SW System with Linked Encoders

Oohama [5] considered the coding problem for correlated sources, where two separate encoders of the SW code are mutually linked as shown in Figure II. We call this compression system *the SWL system* in the meaning of the SW system having the *linkage* of two encoders. Especially, when both switches are closed, we call it *the SWL-I system*. Also, when both switches are open, we call it *the SWL-II system*. First, we define fixed-length coding for the SWL-I system (called *f-SWL-I system*).

Definition 4: A sequence $\{(\varphi_n^{(11)}, \varphi_n^{(12)}, \varphi_n^{(21)}, \varphi_n^{(22)}, \varphi_n^{-1})\}_{n=1}^{\infty}$ of sets of encoders and a decoder is called a *f-SWL-I code*, if the encoders $\varphi_n^{(11)} : \mathcal{X}^n \to \mathcal{M}_n^{(11)}$, $\varphi_n^{(12)} : \mathcal{X}^n \times \mathcal{M}_n^{(21)} \to \mathcal{M}_n^{(12)}$, $\varphi_n^{(21)} : \mathcal{Y}^n \to \mathcal{M}_n^{(21)}$, $\varphi_n^{(22)} : \mathcal{Y}^n \times \mathcal{M}_n^{(11)} \to \mathcal{M}_n^{(22)}$ and the decoder $\varphi_n^{-1} : \mathcal{M}_n^{(11)} \times \mathcal{M}_n^{(12)} \times \mathcal{M}_n^{(21)} \times \mathcal{M}_n^{(22)} \to \mathcal{X}^n \times \mathcal{Y}^n$ satisfy

$$\begin{aligned} \lim_{n\to\infty} \Pr\{&\varphi_n^{-1}(\varphi_n^{(11)}(X^n), \varphi_n^{(12)}(X^n, \varphi_n^{(21)}(Y^n)), \\ &\varphi_n^{(21)}(Y^n), \varphi_n^{(22)}(Y^n, \varphi_n^{(11)}(X^n))) \neq (X^n, Y^n)\} = 0 \quad (1) \end{aligned}$$

where $\mathcal{M}_n^{(ij)} \triangleq \{1, 2, \cdots, M_n^{(ij)}\}$ for $i, j = 1, 2$.

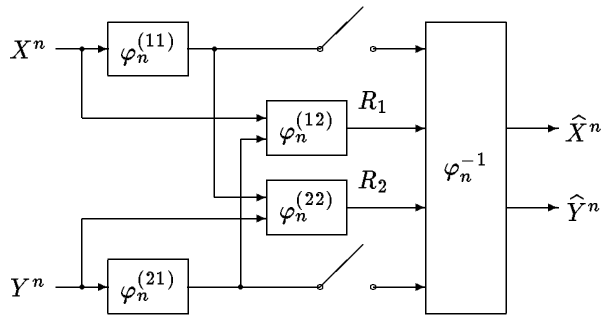

Figure 1: SW system with linked encoders

Definition 5: A rate pair (R_1, R_2) is *admissible for the f-SWL-I system*, if there exists a f-SWL-I code which satisfies

$$\left.\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \log M_n^{(11)} = 0,\ \limsup_{n\to\infty} \frac{1}{n} \log M_n^{(21)} = 0, \\ &\limsup_{n\to\infty} \frac{1}{n} \log M_n^{(12)} \leq R_1,\ \limsup_{n\to\infty} \frac{1}{n} \log M_n^{(22)} \leq R_2. \end{aligned}\right\} \quad (2)$$

Definition 6 (The f-SWL-I region): The f-SWL-I region is defined as a set of (R_1, R_2) which is admissible for the f-SWL-I system.

Similarly, we define the fixed-length coding for the SWL-II system.

Definition 7: A sequence $\{(\varphi_n^{(11)}, \varphi_n^{(12)}, \varphi_n^{(21)}, \varphi_n^{(22)}, \varphi_n^{-1})\}_{n=1}^{\infty}$ of sets of encoders and a decoder is called a *fixed-length SWL-II code*, if the encoders defined in Definition 4 and the decoder $\varphi_n^{-1} : \mathcal{M}_n^{(12)} \times \mathcal{M}_n^{(22)} \to \mathcal{X}^n \times \mathcal{Y}^n$ satisfy

$$\begin{aligned} \lim_{n\to\infty} \Pr\{&\varphi_n^{-1}(\varphi_n^{(12)}(X^n, \varphi_n^{(21)}(Y^n)), \\ &\varphi_n^{(22)}(Y^n, \varphi_n^{(11)}(X^n))) \neq (X^n, Y^n)\} = 0. \end{aligned}$$

Definition 8: A rate pair (R_1, R_2) is *admissible for the f-SWL-II system*, if there exists a f-SWL-II code which satisfies (2).

Definition 9 (The f-SWL-II region): The f-SWL-II region $\mathcal{R}_{SWL-II}(\boldsymbol{X}, \boldsymbol{Y})$ is defined as a set of (R_1, R_2) which is admissible for the f-SWL-II system.

The next theorem clarifies that the region of admissible rate pairs does not expand even if there are mutual linkages between two encoders.

Theorem 2: For any correlated general source $(\boldsymbol{X}, \boldsymbol{Y})$,

$$\mathcal{R}_{SWL-I}(\boldsymbol{X}, \boldsymbol{Y}) = \mathcal{R}_{SWL-II}(\boldsymbol{X}, \boldsymbol{Y}) = \mathcal{R}_{SW}(\boldsymbol{X}, \boldsymbol{Y}).$$

D. Weak Variable-Length Coding

We define the weak variable-length coding for the SWL-I system called *the wv-SWL-I system*).

Definition 10: A sequence $\{(\varphi_n^{(11)}, \varphi_n^{(12)}, \varphi_n^{(21)}, \varphi_n^{(22)}, \varphi_n^{-1})\}_{n=1}^{\infty}$ of encoders and a decoder is called a *wv-SWL-I code*, if the encoders

$$\varphi_n^{(11)} : \mathcal{X}^n \to \mathcal{B}^*, \varphi_n^{(12)} : \mathcal{X}^n \times \varphi_n^{(21)}(\mathcal{Y}^n) \to \mathcal{B}^*,$$
$$\varphi_n^{(21)} : \mathcal{Y}^n \to \mathcal{B}^*, \varphi_n^{(22)} : \mathcal{Y}^n \times \varphi_n^{(11)}(\mathcal{X}^n) \to \mathcal{B}^*,$$

and the decoder $\varphi_n^{-1} : \mathcal{B}^* \times \mathcal{B}^* \times \mathcal{B}^* \times \mathcal{B}^* \to \mathcal{X}^n \times \mathcal{Y}^n$ satisfy the following conditions:

1. The images of $\varphi_n^{(11)}$, $\varphi_n^{(12)}$, $\varphi_n^{(21)}$ and $\varphi_n^{(22)}$ are all prefix sets.
2. $\lim_{n\to\infty} \Pr\{\varphi_n^{-1}(\varphi_n^{(11)}(X^n), \varphi_n^{(12)}(X^n, \varphi_n^{(21)}(Y^n)), \varphi_n^{(21)}(Y^n), \varphi_n^{(22)}(Y^n, \varphi_n^{(11)}(X^n))) \neq (X^n, Y^n)\} = 0.$

Definition 11: A rate pair (R_1, R_2) is *admissible for the wv-SWL-I system*, if there exists a wv-SWL-I code which satisfies

$$\left.\begin{array}{rcl} \limsup_{n\to\infty} \frac{1}{n} E[l(\varphi_n^{(11)}(X^n))] & = & 0, \\ \limsup_{n\to\infty} \frac{1}{n} E[l(\varphi_n^{(21)}(Y^n))] & = & 0, \\ \limsup_{n\to\infty} \frac{1}{n} E\left[l(\varphi_n^{(12)}(X^n, \varphi_n^{(21)}(Y^n)))\right] & \leq & R_1, \\ \limsup_{n\to\infty} \frac{1}{n} E\left[l(\varphi_n^{(22)}(Y^n, \varphi_n^{(11)}(X^n)))\right] & \leq & R_2, \end{array}\right\} \quad (3)$$

where $E[\cdot]$ denotes the expected value and $l : \mathcal{B}^* \to \{0, 1, \cdots\}$ denotes the length function.

Definition 12 (The wv-SWL-I region): The wv-SWL-I region $\mathcal{R}^*_{SWL-I}(\boldsymbol{X}, \boldsymbol{Y})$ is defined as a set of (R_1, R_2) which is admissible for the wv-SWL-I system.

Similarly, we also define a weak variable-length coding for the SWL-II system (called *the wv-SWL-II system*).

Definition 13: A sequence $\{(\varphi_n^{(11)}, \varphi_n^{(12)}, \varphi_n^{(21)}, \varphi_n^{(22)}, \varphi_n^{-1})\}_{n=1}^{\infty}$ of sets is called a *wv-SWL-II code*, if the encoders defined in Definition 10 and the decoder $\varphi_n^{-1} : \mathcal{B}^* \times \mathcal{B}^* \to \mathcal{X}^n \times \mathcal{Y}^n$ satisfy the following conditions:

1. The images of $\varphi_n^{(11)}$, $\varphi_n^{(12)}$, $\varphi_n^{(21)}$ and $\varphi_n^{(22)}$ are all prefix sets.
2. $\lim_{n\to\infty} \Pr\{\varphi_n^{-1}(\varphi_n^{(12)}(X^n, \varphi_n^{(21)}(Y^n)), \varphi_n^{(22)}(Y^n, \varphi_n^{(11)}(X^n))) \neq (X^n, Y^n)\} = 0.$

Definition 14: A rate pair (R_1, R_2) is *admissible for the wv-SWL-II system*, if there exists a wv-SWL-II code which satisfies (3).

Definition 15: (The wv-SWL-II region) The wv-SWL-II region $\mathcal{R}^*_{SWL-II}(\boldsymbol{X}, \boldsymbol{Y})$ is defined as a set of (R_1, R_2) which is admissible for the wv-SWL-II system.

III. Main Results

Before we describe our main result, we impose an assumption for correlated mixed sources.

Assumption: A correlated mixed source $(\boldsymbol{X}, \boldsymbol{Y})$ satisfies at least one of the following conditions ① – ③:

① $H(\boldsymbol{X}_{(1)}) \neq H(\boldsymbol{X}_{(2)})$
② $H(\boldsymbol{Y}_{(1)}) \neq H(\boldsymbol{Y}_{(2)})$
③ $H(\boldsymbol{X}_{(1)}, \boldsymbol{Y}_{(1)}) \neq H(\boldsymbol{X}_{(2)}, \boldsymbol{Y}_{(2)})$

This assumption indicates that two ergodic sources $(\boldsymbol{X}_{(1)}, \boldsymbol{Y}_{(1)})$ and $(\boldsymbol{X}_{(2)}, \boldsymbol{Y}_{(2)})$ can be discriminated by the entropy rate.

The next theorem is our main result.

Theorem 3: If $(\boldsymbol{X}, \boldsymbol{Y})$ is a correlated mixed source, then

$$\begin{aligned} \mathcal{R}^*_{SWL-I}(\boldsymbol{X}, \boldsymbol{Y}) = \Big\{ & (R_1, R_2) : \\ & R_1 \geq \alpha H(\boldsymbol{X}_{(1)}|\boldsymbol{Y}_{(1)}) + (1-\alpha) H(\boldsymbol{X}_{(2)}|\boldsymbol{Y}_{(2)}), \\ & R_2 \geq \alpha H(\boldsymbol{Y}_{(1)}|\boldsymbol{X}_{(1)}) + (1-\alpha) H(\boldsymbol{Y}_{(2)}|\boldsymbol{X}_{(2)}), \\ & R_1 + R_2 \geq \alpha H(\boldsymbol{X}_{(1)}, \boldsymbol{Y}_{(1)}) + (1-\alpha) H(\boldsymbol{X}_{(2)}, \boldsymbol{Y}_{(2)}) \Big\}. \end{aligned}$$

According to Theorem 2, Theorem 3 and Corollary 1, we conclude that the wv-SWL-I region strictly includes the f-SWL-I region, i.e. $\mathcal{R}^*_{SWL-I}(\boldsymbol{X}, \boldsymbol{Y}) \supset \mathcal{R}_{SWL-I}(\boldsymbol{X}, \boldsymbol{Y})$ for any correlated mixed source $(\boldsymbol{X}, \boldsymbol{Y})$ satisfying Assumption. This implies that wv-SWL-I code can achieve strictly lower coding rate than f-SWL-I code. However, for any correlated ergodic source $(\boldsymbol{X}, \boldsymbol{Y})$, we have $\mathcal{R}^*_{SWL-I}(\boldsymbol{X}, \boldsymbol{Y}) = \mathcal{R}_{SWL-I}(\boldsymbol{X}, \boldsymbol{Y})$. Hence, we cannot improve the coding rate for ergodic sources even if we employ wv-SWL-I codes instead of f-SWL-I codes.

The next theorem clarifies that wv-SWL-II codes can achieve the same coding rate as wv-SWL-I codes even if the coding system is restricted.

Theorem 4: If $(\boldsymbol{X}, \boldsymbol{Y})$ is a correlated mixed source, then

$$\mathcal{R}^*_{SWL-II}(\boldsymbol{X}, \boldsymbol{Y}) = \mathcal{R}^*_{SWL-I}(\boldsymbol{X}, \boldsymbol{Y}).$$

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