

Multiterminal source coding for cascading and feedback refinement systems

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Abstract— Lossy coding problems are investigated for some communication systems in the presence of cascading and/or feedback information channels from decoders so as to refine reproduction messages. This framework provides different types of refinement structures from so-called successive refinement. Three different types of communication systems are considered, i.e. refinement systems in the presence of a cascading channel, a feedback channel, and both channels. Outer and inner bounds of achievable rate-distortion regions for those problems are obtained.

Keywords— multiterminal source coding, scalable coding, side information, cascading, refinement, feedback

1 Introduction

Scalable coding problems such as successive refinement [1, 2] and multiresolution coding [3, 4] has attracted considerable attention in the last few decades. In the original setting of successive refinement, the encoder is assumed to operate in two stages, where in the first stage the encoder transmits a source description over a relatively rate-limited channel, and in the second stage the encoder transmits a secondary description to provide a more accurate reproduction of the source. Some extended coding problems have also been investigated. Steinberg and Merhav considered a successive refinement problem in the presence of correlated side information at the decoder [5] and also considered its extension to joint source-channel coding [6].

In contrast, this report discusses other types of refinement structures, which involve cascading and/or feedback transmission from decoders. Three types of communication systems are explored.

Figure 1 shows a block diagram of the first coding problem. There are two encoders, both of which observe a message from a source X and deliver the encoded messages to corresponding decoders. These two decoders have access to correlated side information Y . A cascading channel runs from one decoder to the other, and part of the information obtained at the first stage encoder is sent via the channel to refine a reproduced message. The cascading concept was originally presented by Yamamoto [7], who extended the concept

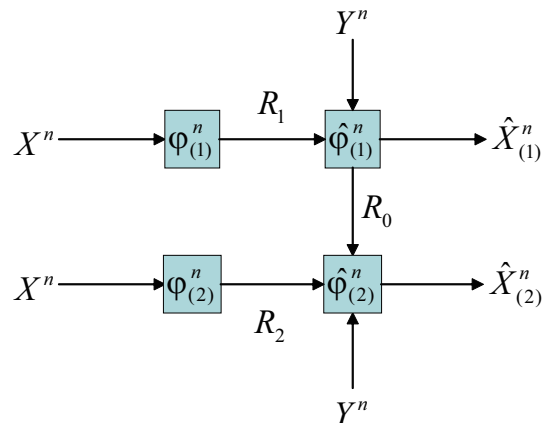


Figure 1: Refinement coding system with cascading channel

to the triangular communication system [8]. The coding problem indicated in Figure 1 is an extended version of the triangular communication system, where correlated side information Y is additionally provided to the decoders.

Figure 2 shows a block diagram of the second coding problem, which is quite similar to the first except for the channel from the first decoder. The channel delivers feedback information to the second encoder. Schalkwijk and Bluestein [9] originally investigated a channel coding problem in which there is a feedback channel from the decoder back to the encoder. This work triggered studies of source coding problems with feedback channels. Other analogous coding problems considering feedback channels were studied by Omura [10] and Butman [11]. Recently, Yang et al. [12] investigated a source coding problem with a feedback channel and side information. Although the coding problem investigated in this paper is similar to Yang's work, lossy coding problems are also considered.

Figure 3 shows a block diagram of the third coding problem, which involves cascading and feedback channels. It should be noted that a similar framework can be seen in a joint source-channel coding setting [6]. This communication system models a certain type of information retrieval with index structures. The details will be provided in a future article [13].

Rate-distortion problems are considered in these settings, and outer and inner bounds of the achievable rate-distortion regions are clarified. These bounds

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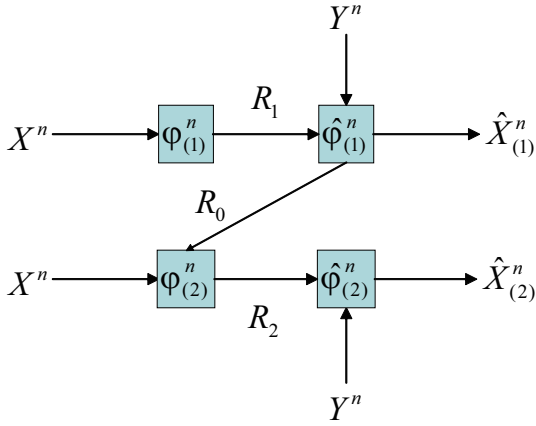


Figure 2: Refinement system with feedback channel

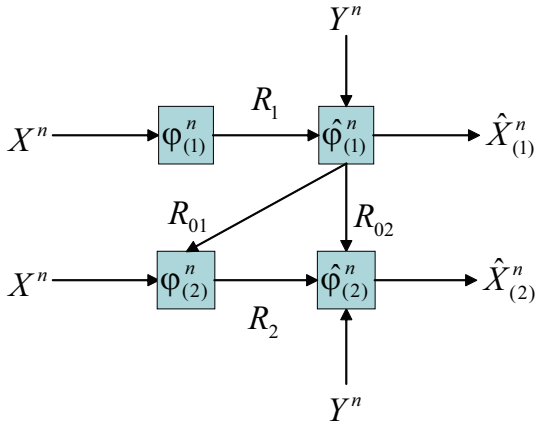


Figure 3: Refinement system with cascading and feedback channels

coincide for some special cases.

2 Definitions and notations

Let \mathcal{X} and \mathcal{Y} be finite sets, $|\mathcal{X}|$ be the cardinality of \mathcal{X} and $\mathcal{I}_M = \{1, 2, \dots, M\}$. A member of \mathcal{X}^n is written as $x^n = (x_1, x_2, \dots, x_n)$, and substrings of x^n are written as $x_i^j = (x_i, x_{i+1}, \dots, x_j)$ for $i \leq j$. When the dimension is clear from the content, vectors will be denoted by boldface letters, i.e., $\mathbf{x} \in \mathcal{X}^n$. $\mathcal{M}(\mathcal{X})$ denotes the set of all probability distributions on \mathcal{X} . A discrete memoryless source (\mathcal{X}, P_X) is an infinite sequence $\mathbf{X} = \{X_i\}_{i=1}^\infty$ of independent copies of a random variable X taking values in \mathcal{X} with a generic distribution $P_X \in \mathcal{M}(\mathcal{X})$. Similarly, the source $(\mathcal{X} \times \mathcal{Y}, P_{XY})$ is an infinite sequence of independent copies of a pair of random variables (X, Y) taking values in $\mathcal{X} \times \mathcal{Y}$ with generic joint distribution $P_{XY} \in \mathcal{M}(\mathcal{X} \times \mathcal{Y})$. Also, $\mathcal{M}(\mathcal{X}|P_Y)$ denotes the set of all probability distributions on \mathcal{X} given a distribution $P_Y \in \mathcal{M}(\mathcal{Y})$, namely each member of $\mathcal{M}(\mathcal{X}|P_Y)$ is characterized by $P_{XY} \in \mathcal{M}(\mathcal{X} \times \mathcal{Y})$ as $P_{XY} = P_{X|Y}P_Y$. We will denote a source (\mathcal{X}, P_X) by referring to its generic distribution P_X or random variable X . For a correlated source

(X, Y) , $H(X|Y)$ and $I(X; Y)$ denote a conditional entropy of X given Y and mutual information shared by X and Y , respectively. Similarly, for a correlated source (X, Y, Z) , $I(X; Y|Z)$ denotes conditional mutual information shared by X and Y given Z . A similar convention is used for other random variables and vectors. In the following, all bases of exponentials and logarithms are set at 2. Let $\hat{\mathcal{X}}$ stand for a reconstruction alphabet, and let $\Delta : \mathcal{X} \times \hat{\mathcal{X}} \rightarrow [0, \infty)$ be a single-letter distortion function. The vector distortion function is defined in the usual way, i.e. $\Delta_X^n(\mathbf{x}, \hat{\mathbf{x}}) = \frac{1}{n} \sum_{k=1}^n \Delta_X(x_k, \hat{x}_k)$.

3 Refinement with cascading channels

At first, a *cascading refinement coding system* is considered, which involves a refining structure with a cascading channel from the first decoder to the second decoder.

3.1 Problem formulation

Definition 1. (CR (Cascading Refinement) code) A set $(\varphi_{(0)}^n, \varphi_{(1)}^n, \varphi_{(2)}^n, \hat{\varphi}_{(1)}^n, \hat{\varphi}_{(2)}^n)$ of encoders and decoders is a CR code $(n, M_n^{(0)}, M_n^{(1)}, M_n^{(2)}, \rho_n^{(1)}, \rho_n^{(2)})$ for the source (X, Y) if and only if

$$\begin{aligned} \varphi_{(1)}^n &: \mathcal{X}^n \rightarrow \mathcal{I}_{M_n^{(1)}} \\ \varphi_{(0)}^n &: \mathcal{Y}^n \times \mathcal{I}_{M_n^{(1)}} \rightarrow \mathcal{I}_{M_n^{(0)}} \\ \varphi_{(2)}^n &: \mathcal{X}^n \rightarrow \mathcal{I}_{M_n^{(2)}} \\ \hat{\varphi}_{(1)}^n &: \mathcal{I}_{M_n^{(1)}} \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n, \\ \hat{\varphi}_{(2)}^n &: \mathcal{I}_{M_n^{(0)}} \times \mathcal{I}_{M_n^{(2)}} \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n, \end{aligned}$$

where

$$\begin{aligned} \rho_n^{(i)} &= E \left[\Delta^n(X^n, \hat{X}_{(i)}^n) \right], \quad (i = 1, 2) \\ A_n^{(1)} &= \varphi_{(1)}^n(X^n), \quad A_n^{(0)} = \varphi_{(0)}^n(A_n^{(1)}, Y^n), \\ A_n^{(2)} &= \varphi_{(2)}^n(X^n), \\ \hat{X}_{(1)}^n &= \hat{\varphi}_{(1)}^n(A_n^{(1)}, Y^n), \\ \hat{X}_{(2)}^n &= \hat{\varphi}_{(2)}^n(A_n^{(0)}, A_n^{(2)}, Y^n). \end{aligned}$$

Definition 2. (CR-achievable rate triad)

(R_0, R_1, R_2) is a CR-achievable rate triad of the source (X, Y) for a given distortion pair (D_1, D_2) if and only if there exists a sequence of CR codes $\{(n, M_n^{(0)}, M_n^{(1)}, M_n^{(2)}, \rho_n^{(1)}, \rho_n^{(2)})\}_{n=1}^\infty$ for the source (X, Y) such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log M_n^{(i)} &\leq R_i, \quad (i = 0, 1, 2) \\ \limsup_{n \rightarrow \infty} \rho_n^{(j)} &\leq D_j. \quad (j = 1, 2) \end{aligned}$$

Definition 3. (CR-achievable rate region)

$\mathcal{R}_c(X, Y|D_1, D_2) = \{(R_0, R_1, R_2) : (R_0, R_1, R_2) \text{ is a CR-achievable rate triad of } (X, Y) \text{ for } (D_1, D_2)\}$.

3.2 Statement of results

Theorem 1. (Coding theorem of CR code)

$$\begin{aligned} \mathcal{R}_c(X, Y|D_1, D_2) &\subseteq \{(R_0, R_1, R_2) : \\ R_1 &\geq I(X; UV|Y), \\ R_0 &\geq I(X; V|Y), \\ R_2 &\geq I(X; W|VY)\} \quad (\text{outer bound}) \end{aligned}$$

where the random variables U , V and W whose alphabets are \mathcal{U} , \mathcal{V} and \mathcal{W} , respectively, are selected such that

- the alphabet sizes are bounded as

$$\begin{aligned} |\mathcal{U}| &\leq |\mathcal{X}| + 1, \\ |\mathcal{W}| &\leq |\mathcal{U} \times \mathcal{X}| + 1, \\ |\mathcal{V}| &\leq |\mathcal{U} \times \mathcal{W} \times \mathcal{X} \times \mathcal{Y}| + 4, \end{aligned}$$

- $UW \rightarrow X \rightarrow Y$ forms a Markov chain,
- there exist functions $\phi_{(1)} : \mathcal{U} \times \mathcal{V} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}$ and $\phi_{(2)} : \mathcal{V} \times \mathcal{W} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}$ which satisfy

$$\begin{aligned} D_1 &\geq E[\Delta(X, \phi_{(1)}(U, V, Y))], \\ D_2 &\geq E[\Delta(X, \phi_{(2)}(V, W, Y))]. \end{aligned}$$

An inner bound is obtained with the same functional forms, while the Markov chain is replaced as $UVW \rightarrow X \rightarrow Y$.

From Theorem 1, we can obtain the following properties.

Corollary 1. (Compatibility with a known result)

If side information Y is not available at both of two decoders, the outer bound indicated in Theorem 1 coincides with the inner bound, i.e.

$$\begin{aligned} \mathcal{R}_c(X|D_1, D_2) &= \{(R_0, R_1, R_2) : \\ R_1 &\geq I(X; \hat{X}_{(1)}|V), \\ R_0 &\geq I(X; V), \\ R_2 &\geq I(X; \hat{X}_{(2)}|V)\} \end{aligned}$$

where the random variable V taking a value in \mathcal{V} is selected such that $|\mathcal{V}| \leq |\mathcal{X}| + 2$. This rate region coincides with the one indicated by Yamamoto [8].

In the above discussions, we have considered only two-stage refinement. However, they can be easily extended to communication systems with multi-stage refinement. Let us consider N pairs $\{\varphi_{(i)}, \hat{\varphi}_{(i)}\}_{i \in \mathcal{I}_N}$ of encoders and decoders, and $(N-1)$ cascading encoders $\{\varphi_{(0j)}\}_{j \in \mathcal{I}_{N-1}}$. Let $\mathbf{D} = \{D_i\}_{i \in \mathcal{I}_N}$ be a set of distortion criteria, each of which corresponds to the decoder $\hat{\varphi}_{(i)}$. We define the CR-achievable rate region $\mathcal{R}_c(X, Y|\mathbf{D})$ of the source (X, Y) for given distortion criteria \mathbf{D} in the same way as for the two-stage refinement system, where R_i ($i \in \mathcal{I}_N$) corresponds to the rate of the encoder $\varphi_{(i)}$, and R_{0j} ($j \in \mathcal{I}_{N-1}$) corresponds to the rate of the cascading encoder $\varphi_{(0j)}$. Let $U^{(i)}$ ($i \in \mathcal{I}_N$) be an auxiliary random variable that takes a value in a

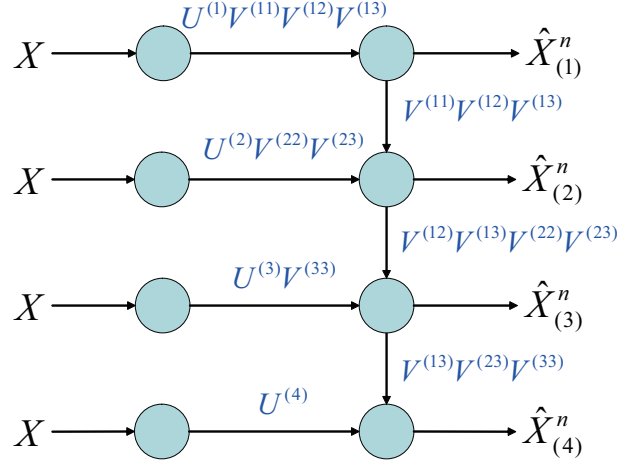


Figure 4: Diagram of 4-stage CR code (without side information for simplicity)

finite set $\mathcal{U}^{(i)}$, and $V^{(i,j)}$ ($i, j \in \mathcal{I}_{N-1}$) be an auxiliary random variable that takes values in a finite set $\mathcal{V}^{(i,j)}$. For some $\mathcal{S} \subseteq \mathcal{I}_N$ and $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{I}_{N-1}$, let us define

$$\begin{aligned} \mathcal{U}^{(\mathcal{S})} &= \{U^{(i)} : i \in \mathcal{S}\}, \\ \mathcal{U} &= \mathcal{U}^{(\mathcal{I}_N)}, \\ \mathcal{V}^{(i, \mathcal{T}_2)} &= \{V^{(i,j)} : j \in \mathcal{T}_2\}, \\ \mathcal{V}^{(\mathcal{T}_1, \mathcal{T}_2)} &= \{V^{(i,j)} \in \mathcal{V}^{(i, \mathcal{T}_2)} : i \in \mathcal{T}_1\}, \\ \mathcal{V} &= \bigcup_{k \in \mathcal{I}_{N-1}} \mathcal{V}^{(k, \mathcal{I}_{N-1} - \mathcal{I}_{k-1})}, \\ \mathcal{U}^{(\mathcal{S})} &= \prod_{i \in \mathcal{S}} \mathcal{U}^{(i)}, \\ \mathcal{V}^{(i, \mathcal{T}_2)} &= \prod_{j \in \mathcal{T}_2} \mathcal{V}^{(i,j)}, \\ \mathcal{V}^{(\mathcal{T}_1, \mathcal{T}_2)} &= \prod_{i \in \mathcal{T}_1} \mathcal{V}^{(i, \mathcal{T}_2)}. \end{aligned}$$

Corollary 2. (Coding theorem of CR code with multi-stage cascading)

$$\begin{aligned} \mathcal{R}_c(X, Y|\mathbf{D}) &\subseteq \{(R_{0j}, R_i)_{i \in \mathcal{I}_N, j \in \mathcal{I}_{N-1}} : \\ R_i &\geq I(X; U^{(i)} \mathbf{V}^{(i, \mathcal{I}_{N-1} - \mathcal{I}_{i-1})} | \mathbf{V}^{(\mathcal{I}_{i-1}, \mathcal{I}_{N-1} - \mathcal{I}_{i-2})} Y), \\ R_{0j} &\geq I(X; \mathbf{V}^{(\mathcal{I}_j, \mathcal{I}_{N-1} - \mathcal{I}_{j-1})} | Y)\} \quad (\text{outer bound}) \end{aligned}$$

where the random variables \mathbf{U} and \mathbf{V} whose alphabets are $\mathcal{U}^{(\mathcal{I}_N)}$ and $\prod_{k \in \mathcal{I}_{N-1}} \mathcal{V}^{(k, \mathcal{I}_{N-1} - \mathcal{I}_{k-1})}$, respectively, are selected such that

- the alphabet sizes are bounded as

$$\begin{aligned} |\mathcal{U}^{(i)}| &\leq \left| \mathcal{U}^{(\mathcal{I}_{i-1})} \times \prod_{k=1}^{i-1} \mathcal{V}^{(k, \mathcal{I}_{N-1} - \mathcal{I}_{k-1})} \times \mathcal{X} \right| + 2, \\ &\quad (i \in \mathcal{I}_N), \\ |\mathcal{V}^{(i,j)}| &\leq \dots \end{aligned}$$

$$\leq \left| \mathcal{U}^{(i)} \times \prod_{k=1}^{i-1} \left\{ \mathcal{V}^{(k, \mathcal{I}_{N-1} - \mathcal{I}_{k-1})} \right\} \times \mathcal{V}^{(i, \mathcal{I}_{N-1} - \mathcal{I}_j)} \right. \\ \left. \times \mathcal{X} \times \mathcal{Y} \right| + (3j + 2) \quad (i, j \in \mathcal{I}_{N-1}),$$

- $\mathbf{U} \rightarrow \mathbf{X} \rightarrow \mathbf{Y}$ form Markov chains,
- there exist functions $\{\phi_{(i)} : \mathcal{U}^{(i)} \times \mathcal{V}^{(\mathcal{I}_{i-1}, \mathcal{I}_{N-1} - \mathcal{I}_{i-2})} \times \mathcal{V}^{(i, \mathcal{I}_{N-1} - \mathcal{I}_{i-1})} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}\}_{i \in \mathcal{I}_N}$ which satisfy

$$D_i \geq E[\Delta(X, \phi_{(i)}(U^{(i)}, \mathbf{V}^{(\mathcal{I}_{i-1}, \mathcal{I}_{N-1} - \mathcal{I}_{i-2})}, \mathbf{V}^{(i, \mathcal{I}_{N-1} - \mathcal{I}_{i-1})}, Y))].$$

An inner bound is obtained with the same functional forms, while the Markov chains are replaced as $\mathbf{UV} \rightarrow \mathbf{X} \rightarrow \mathbf{Y}$.

4 Refinement with feedback channels

Next, a *feedback refinement system* is considered, which indicates the refinement system with a feedback channel from the first decoder back to the second decoder.

4.1 Problem formulation

Definition 4. (FR (Feedback Refinement) code)

A set $(\varphi_{(0)}^n, \varphi_{(1)}^n, \varphi_{(2)}^n, \hat{\varphi}_{(1)}^n, \hat{\varphi}_{(2)}^n)$ of encoders and decoders is an FR code $(n, M_n^{(0)}, M_n^{(1)}, M_n^{(2)}, \rho_n^{(1)}, \rho_n^{(2)})$ for the source (X, Y) if and only if

$$\begin{aligned} \varphi_{(1)}^n &: \mathcal{X}^n \rightarrow \mathcal{I}_{M_n^{(1)}} \\ \varphi_{(0)}^n &: \mathcal{I}_{M_n^{(1)}} \times \mathcal{Y}^n \rightarrow \mathcal{I}_{M_n^{(0)}} \\ \varphi_{(2)}^n &: \mathcal{I}_{M_n^{(0)}} \times \mathcal{X}^n \rightarrow \mathcal{I}_{M_n^{(2)}} \\ \hat{\varphi}_{(1)}^n &: \mathcal{I}_{M_n^{(1)}} \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n, \\ \hat{\varphi}_{(2)}^n &: \mathcal{I}_{M_n^{(2)}} \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n, \end{aligned}$$

where

$$\begin{aligned} \rho_n^{(i)} &= E[\Delta^n(X^n, \hat{X}_{(i)}^n)], \quad (i = 1, 2) \\ A_n^{(1)} &= \varphi_{(1)}^n(X^n), \quad A_n^{(0)} = \varphi_{(0)}^n(A_n^{(1)}, Y^n), \\ A_n^{(2)} &= \varphi_{(2)}^n(A_n^{(0)}, X^n), \\ \hat{X}_{(1)}^n &= \hat{\varphi}_{(1)}^n(A_n^{(1)}, Y^n), \quad \hat{X}_{(2)}^n = \hat{\varphi}_{(2)}^n(A_n^{(2)}, Y^n). \end{aligned}$$

Definition 5. (FR-achievable rate triad)

(R_0, R_1, R_2) is an FR-achievable rate triad of the source (X, Y) for a given distortion pair (D_1, D_2) if and only if there exists a sequence of FR codes $\{(n, M_n^{(0)}, M_n^{(1)}, M_n^{(2)}, \rho_n^{(1)}, \rho_n^{(2)})\}_{n=1}^\infty$ for the source (X, Y) such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log M_n^{(i)} &\leq R_i, \quad (i = 0, 1, 2) \\ \limsup_{n \rightarrow \infty} \rho_n^{(j)} &\leq D_j. \quad (j = 1, 2) \end{aligned}$$

Definition 6. (FR-achievable rate region)

$\mathcal{R}_f(X, Y|D_1, D_2) = \{(R_0, R_1, R_2) : (R_0, R_1, R_2) \text{ is an FR-achievable rate triad of } (X, Y) \text{ for } (D_1, D_2)\}$.

4.2 Statement of results

First, we state the main theorem.

Theorem 2. (Coding theorem of FR code)

$$\begin{aligned} \mathcal{R}_f(X, Y|D_1, D_2) &\subseteq \{(R_0, R_1, R_2) : \\ R_1 &\geq I(X; U|Y), \\ R_0 &\geq I(Y; V|UX), \\ R_2 &\geq I(X; W|VY)\} \quad (\text{outer bound}) \end{aligned}$$

where the random variables U, V and W whose alphabets are \mathcal{U}, \mathcal{V} and \mathcal{W} , respectively, are selected such that

- the alphabet sizes are bounded as

$$\begin{aligned} |\mathcal{U}| &\leq |\mathcal{X}| + 2, \\ |\mathcal{V}| &\leq |\mathcal{U} \times \mathcal{Y}| + 1, \\ |\mathcal{W}| &\leq |\mathcal{U} \times \mathcal{V} \times \mathcal{X}| + 1, \end{aligned}$$

- Markov chains

$$\begin{aligned} U &\rightarrow X \rightarrow Y, \\ V &\rightarrow UY \rightarrow X, \\ W &\rightarrow VX \rightarrow UY \end{aligned}$$

are satisfied,

- there exist functions $\phi_{(1)} : \mathcal{U} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}$ and $\phi_{(2)} : \mathcal{W} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}$ which satisfy

$$\begin{aligned} D_1 &\geq E[\Delta(X, \phi_{(1)}(U, Y))], \\ D_2 &\geq E[\Delta(X, \phi_{(2)}(W, Y))]. \end{aligned}$$

An inner bound is obtained in the same functional forms, while the Markov chains are replaced as

$$\begin{aligned} U &\rightarrow X \rightarrow Y, \\ V &\rightarrow Y \rightarrow UX, \\ W &\rightarrow VX \rightarrow UY. \end{aligned}$$

Remark . The Markov condition $V \rightarrow Y \rightarrow UX$ is tight compared with the condition $V \rightarrow UY \rightarrow X$ because $I(V; X|UY) \leq I(V; UX|Y)$.

Remark . The Markov conditions $U \rightarrow X \rightarrow Y, V \rightarrow UY \rightarrow X$ and $W \rightarrow VX \rightarrow UY$ are equivalent to the condition that the joint distribution P_{UVWXY} satisfies

$$\begin{aligned} P_{UVWXY}(u, v, w, x, y) \\ = P_{XY}(x, y)P_{U|X}(u|x)P_{V|UY}(v|u, y)P_{W|VX}(w|v, x). \end{aligned}$$

In a similar manner, the Markov conditions $U \rightarrow X \rightarrow Y, V \rightarrow Y \rightarrow UX$ and $W \rightarrow VX \rightarrow UY$ are equivalent to the condition that the joint distribution P_{UVWXY} satisfies

$$\begin{aligned} P_{UVWXY}(u, v, w, x, y) \\ = P_{XY}(x, y)P_{U|X}(u|x)P_{V|Y}(v|y)P_{W|VX}(w|v, x). \end{aligned}$$

Theorem 2 indicates that feedback information contributes to refining reproduction messages, in contrast to the result for a lossless configuration reported by Yang et al. [12].

From Theorem 2, we can obtain the following properties.

Corollary 3. (Coding theorem for some special cases)
If side information Y is not available at both of two decoders, the outer bound indicated in Theorem 2 coincides with the inner bound, i.e.

$$\begin{aligned} \mathcal{R}_f(X|D_1, D_2) &= \{(R_0, R_1, R_2) : \\ R_1 &\geq I(X; \hat{X}^{(1)}), \\ R_2 &\geq I(X; \hat{X}^{(2)})\}. \end{aligned}$$

This shows that feedback information is of no use for refining in the absence of side information.

The above discussions can be easily extended to communication systems with multi-stage refinement. Let us consider N pairs $\{\varphi_{(i)}, \hat{\varphi}_{(i)}\}_{i \in \mathcal{I}_N}$ of encoders and decoders, and $N - 1$ feedback encoders $\{\varphi_{(0j)}\}_{j \in \mathcal{I}_{N-1}}$. Let $\mathbf{D} = \{D_i\}_{i \in \mathcal{I}_N}$ be a set of distortion criteria, each of which corresponds to the decoder $\hat{\varphi}_{(i)}$. We define the FR-achievable rate region $\mathcal{R}_f(X, Y|\mathbf{D})$ of the source (X, Y) for given distortion criteria \mathbf{D} in the same way as for the two-stage refinement system, where R_i ($i \in \mathcal{I}_N$) corresponds to the rate of the encoder $\varphi_{(i)}$, and $R_{(0j)}$ ($j \in \mathcal{I}_{N-1}$) corresponds to the rate of the feedback encoder $\varphi_{(0j)}$. Let $U^{(i)}$ ($i \in \mathcal{I}_N$) and $V^{(j)}$ ($j \in \mathcal{I}_{N-1}$) be auxiliary random variables that take values in finite sets $\mathcal{U}^{(i)}$ and $\mathcal{V}^{(j)}$, respectively. For some $\mathcal{S} \subseteq \mathcal{I}_N$ and $\mathcal{T} \subseteq \mathcal{I}_{N-1}$, let us define

$$\begin{aligned} \mathbf{U}^{(\mathcal{S})} &= \{U^{(i)} : i \in \mathcal{S}\}, \\ \mathbf{U} &= \mathbf{U}^{(\mathcal{I}_N)}, \\ \mathbf{V}^{(\mathcal{T})} &= \{V^{(j)} : j \in \mathcal{T}\}, \\ \mathbf{V} &= \mathbf{V}^{(\mathcal{I}_{N-1})}, \\ \mathcal{U}^{(\mathcal{I}_N)} &= \prod_{i \in \mathcal{I}_N} \mathcal{U}^{(i)}, \\ \mathcal{V}^{(\mathcal{I}_{N-1})} &= \prod_{j \in \mathcal{I}_{N-1}} \mathcal{V}^{(j)}. \end{aligned}$$

Corollary 4. (Coding theorem of FR code with multi-stage feedback)

$$\begin{aligned} \mathcal{R}_f(X, Y|\mathbf{D}) &\subseteq \{(R_{0j}, R_i)_{i \in \mathcal{I}_N, j \in \mathcal{I}_{N-1}} : \\ R_{0j} &\geq I(Y; V^{(j)}|U^{(i)}X), \\ R_i &\geq I(X; U^{(i)}|V^{(i-1)}Y)\} \quad (\text{outer bound}) \end{aligned}$$

where the random variables \mathbf{U} and \mathbf{V} whose alphabets are $\mathcal{U}^{(\mathcal{I}_N)}$ and $\mathcal{V}^{(\mathcal{I}_{N-1})}$, respectively, are selected such that

- the alphabet sizes are bounded as

$$\begin{aligned} |\mathcal{U}^{(i)}| &\leq |\mathcal{U}^{(\mathcal{I}_{i-1})} \times \mathcal{V}^{(\mathcal{I}_{i-1})} \times \mathcal{X}| + 2, \quad (i \in \mathcal{I}_N) \\ |\mathcal{V}^{(i)}| &\leq |\mathcal{U}^{(\mathcal{I}_i)} \times \mathcal{V}^{(\mathcal{I}_{i-1})} \times \mathcal{Y}| + 1, \quad (i \in \mathcal{I}_{N-1}) \end{aligned}$$

- the following Markov chains are satisfied:

$$\begin{aligned} U^{(i)} &\rightarrow V^{(i-1)}X \rightarrow \mathbf{U}^{(\mathcal{I}_{i-1})}\mathbf{V}^{(\mathcal{I}_{i-2})}Y, \quad (i \in \mathcal{I}_N) \\ V^{(i)} &\rightarrow U^{(i)}Y \rightarrow \mathbf{U}^{(\mathcal{I}_{i-1})}\mathbf{V}^{(\mathcal{I}_{i-1})}X, \quad (i \in \mathcal{I}_{N-1}) \end{aligned}$$

- there exist functions $\{\phi_{(i)} : \mathcal{U}^{(i)} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}\}_{i \in \mathcal{I}_N}$ which satisfy

$$D_i \geq E \left[\Delta(X, \phi_{(i)}(U^{(i)}, Y)) \right].$$

An inner bound is obtained with the same functional forms, while the Markov chains are replaced as

$$\begin{aligned} U^{(i)} &\rightarrow V^{(i-1)}X \rightarrow \mathbf{U}^{(\mathcal{I}_{i-1})}\mathbf{V}^{(\mathcal{I}_{i-2})}Y, \quad (i \in \mathcal{I}_N) \\ V^{(i)} &\rightarrow Y \rightarrow \mathbf{U}^{(\mathcal{I}_i)}\mathbf{V}^{(\mathcal{I}_{i-1})}X. \quad (i \in \mathcal{I}_{N-1}) \end{aligned}$$

The above corollary coincides with the result reported by Kaspi [14].

5 Refinement with cascading and feedback channels

Lastly, a *cascading and feedback refinement system* is considered, which denotes a refinement system with a cascading channel from the first decoder to the second decoder and a feedback channel from the first decoder to the second decoder.

5.1 Problem formulation

Definition 7. (CFR (Cascading and Feedback Refinement) code)

A set $(\varphi_{(01)}^n, \varphi_{(02)}^n, \varphi_{(1)}^n, \varphi_{(2)}^n, \hat{\varphi}_{(1)}^n, \hat{\varphi}_{(2)}^n)$ of encoders and decoders is a CFR code $(n, M_n^{(01)}, M_n^{(02)}, M_n^{(1)}, M_n^{(2)}, \rho_n^{(1)}, \rho_n^{(2)})$ for the source (X, Y) if and only if

$$\begin{aligned} \varphi_{(1)}^n &: \mathcal{X}^n \rightarrow \mathcal{I}_{M_n^{(1)}} \\ \varphi_{(01)}^n &: \mathcal{I}_{M_n^{(1)}} \times \mathcal{Y}^n \rightarrow \mathcal{I}_{M_n^{(01)}} \\ \varphi_{(02)}^n &: \mathcal{I}_{M_n^{(1)}} \times \mathcal{Y}^n \rightarrow \mathcal{I}_{M_n^{(02)}} \\ \varphi_{(2)}^n &: \mathcal{I}_{M_n^{(01)}} \times \mathcal{X}^n \rightarrow \mathcal{I}_{M_n^{(2)}} \\ \hat{\varphi}_{(1)}^n &: \mathcal{I}_{M_n^{(1)}} \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n, \\ \hat{\varphi}_{(2)}^n &: \mathcal{I}_{M_n^{(02)}} \times \mathcal{I}_{M_n^{(2)}} \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n, \end{aligned}$$

where

$$\begin{aligned} \rho_n^{(i)} &= E \left[\Delta^n(X^n, \hat{X}_{(i)}^n) \right], \quad (i = 1, 2) \\ A_n^{(1)} &= \varphi_{(1)}^n(X^n), \quad A_n^{(01)} = \varphi_{(01)}^n(A_n^{(1)}, Y^n), \\ A_n^{(02)} &= \varphi_{(02)}^n(A_n^{(1)}, Y^n), \quad A_n^{(2)} = \varphi_{(2)}^n(A_n^{(01)}, X^n), \\ \hat{X}_{(1)}^n &= \hat{\varphi}_{(1)}^n(A_n^{(1)}, Y^n), \quad \hat{X}_{(2)}^n = \hat{\varphi}_{(2)}^n(A_n^{(01)}, A_n^{(2)}, Y^n). \end{aligned}$$

Definition 8. (CFR-achievable rate quadruplet)

$(R_{01}, R_{02}, R_1, R_2)$ is a CFR-achievable rate quadruplet of the source (X, Y) for a given distortion pair (D_1, D_2) if and only if there exists a sequence of CFR codes $\{(n, M_n^{(01)}, M_n^{(02)}, M_n^{(1)}, M_n^{(2)}, \rho_n^{(1)}, \rho_n^{(2)})\}_{n=1}^{\infty}$ for the source (X, Y) such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log M_n^{(i)} \leq R_i, \quad (i = 01, 02, 1, 2)$$

$$\limsup_{n \rightarrow \infty} \rho_n^{(j)} \leq D_j. \quad (j = 1, 2)$$

Definition 9. (CFR-achievable rate region)

$$\mathcal{R}_{cf}(X, Y|D_1, D_2) = \{(R_{01}, R_{02}, R_1, R_2) : \\ (R_{01}, R_{02}, R_1, R_2) \text{ is a CFR-achievable rate} \\ \text{quadruplet of } (X, Y) \text{ for } (D_1, D_2)\}.$$

5.2 Statement of results

First, we state the main theorem.

Theorem 3. (Coding theorem of CFR code)

$$\mathcal{R}_{cf}(X, Y|D_1, D_2) \subseteq \{(R_{01}, R_{02}, R_1, R_2) : \\ R_1 \geq I(X; UV^{(2)}|Y), \\ R_{01} \geq I(Y; V^{(1)}|UV^{(2)}X), \\ R_{02} \geq I(X; V^{(2)}|Y), \\ R_2 \geq I(X; W|V^{(1)}V^{(2)}Y)\} \quad (\text{outer bound})$$

where the random variables $U, V^{(1)}, V^{(2)}$ and W whose alphabets are $\mathcal{U}, \mathcal{V}^{(1)}, \mathcal{V}^{(2)}$ and \mathcal{W} , respectively, are selected such that

- the alphabet sizes are bounded as

$$\begin{aligned} |\mathcal{U}| &\leq |\mathcal{X}| + 2, \\ |\mathcal{V}^{(2)}| &\leq |\mathcal{U} \times \mathcal{X} \times \mathcal{Y}| + 5, \\ |\mathcal{V}^{(1)}| &\leq |\mathcal{U} \times \mathcal{V}^{(2)} \times \mathcal{Y}| + 1, \\ |\mathcal{W}| &\leq |\mathcal{U} \times \mathcal{V}^{(1)} \times \mathcal{V}^{(2)} \times \mathcal{X}| + 1, \end{aligned}$$

- Markov chains

$$\begin{aligned} U &\rightarrow X \rightarrow Y, \\ V^{(1)} &\rightarrow UV^{(2)}Y \rightarrow X, \\ W &\rightarrow V^{(1)}V^{(2)}X \rightarrow UY \end{aligned}$$

are satisfied,

- there exist functions $\phi_{(1)} : \mathcal{U} \times \mathcal{V}^{(2)} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}$ and $\phi_{(2)} : \mathcal{W} \times \mathcal{V}^{(2)} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}$ which satisfy

$$\begin{aligned} D_1 &\geq E \left[\Delta(X, \phi_{(1)}(U, V^{(2)}, Y)) \right], \\ D_2 &\geq E \left[\Delta(X, \phi_{(2)}(W, V^{(2)}, Y)) \right]. \end{aligned}$$

An inner bound is obtained in the same functional forms, while the Markov chains are replaced as

$$\begin{aligned} UV^{(2)} &\rightarrow X \rightarrow Y, \\ V^{(1)} &\rightarrow V^{(2)}Y \rightarrow UX, \\ W &\rightarrow V^{(1)}V^{(2)}X \rightarrow UY. \end{aligned}$$

Remark . The Markov conditions

$$\begin{aligned} U &\rightarrow \hat{X} \rightarrow Y, \\ V^{(1)} &\rightarrow UV^{(2)}Y \rightarrow X, \\ W &\rightarrow V^{(1)}V^{(2)}X \rightarrow UY, \end{aligned}$$

are equivalent to the condition that the joint distribution $P_{UV^{(1)}V^{(2)}WXY}$ satisfies

$$\begin{aligned} &P_{UV^{(1)}V^{(2)}WXY}(u, v^{(1)}, v^{(2)}, w, x, y) \\ &= P_{XY}(x, y)P_{U|X}(u|x)P_{V^{(2)}|UXY}(v^{(2)}|u, x, y) \\ &P_{V^{(1)}|UV^{(2)}Y}(v^{(1)}|u, v^{(2)}, y) \\ &P_{W|V^{(1)}V^{(2)}X}(w|v^{(1)}, v^{(2)}, x). \end{aligned}$$

Also, the Markov conditions

$$\begin{aligned} U &\rightarrow X \rightarrow Y, \\ V^{(1)} &\rightarrow V^{(2)}Y \rightarrow UX, \\ W &\rightarrow V^{(1)}V^{(2)}X \rightarrow UY. \end{aligned}$$

are equivalent to the condition that the joint distribution $P_{UV^{(1)}V^{(2)}WXY}$ satisfies

$$\begin{aligned} &P_{UV^{(1)}V^{(2)}WXY}(u, v^{(1)}, v^{(2)}, w, x, y) \\ &= P_{XY}(x, y)P_{U|X}(u|x)P_{V^{(2)}|UXY}(v^{(2)}|u, x, y) \\ &P_{V^{(1)}|V^{(2)}Y}(v^{(1)}|v^{(2)}, y) \\ &P_{W|V^{(1)}V^{(2)}X}(w|v^{(1)}, v^{(2)}, x) \end{aligned}$$

6 Proof of theorems

The proofs will be provided in a technical report [15].

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