

# Bounds on Classical Capacity of LOCC Quantum Channels

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# Talk Preview

I. Introduction

II. Results

III. Proofs

IV. Discussions

V. Conclusions

# Introduction

## Motivation

### Use / Role of multiparty entanglement in Quantum Communication

- In [1] multiparty entanglement given operational interpretation: entanglement bounds how well we can discriminate states by LOCC
- But, what can we do with this?
- Here, we consider using this in quantum communication
  - Idea: encode information in states, see how LOCC access to information is limited by entanglement

# Introduction

## Why access information by LOCC?

Could be many uses, here are a few

- 1) Environment(s) assisted channel capacity
- 2) Distribution of secret (secret sharing/ data hiding)
- 3) Network of Quantum Computers

# Introduction

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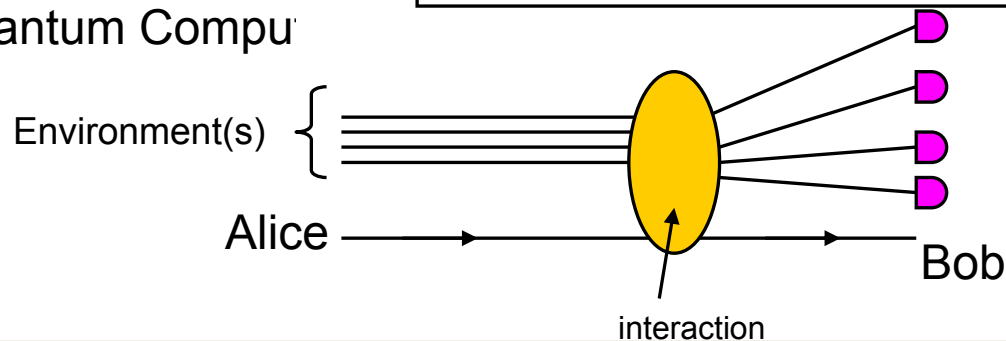
### 1) Environment(s) assisted channel capacity \*

### 2) Distribution of secret (secret

### 3) Network of Quantum Compu

-Channel gets entangled to the environment.

-Measure environment, and “correct”, using LOCC



\* Hayden & King quant-ph/0409026, Winter quant-ph/0507045

# Introduction

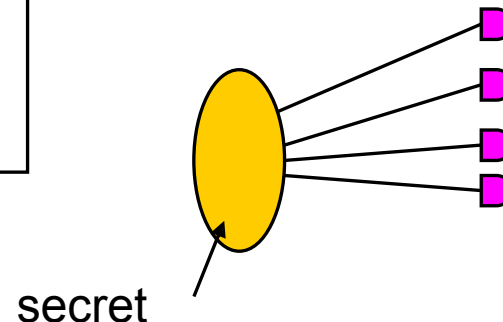
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- 2) Distribution of secret (secret sharing/ data hiding)**

3) Netw

$$C_{\text{LOCC}} < C_{\text{Global}}$$
$$I_{\text{secret}} := I_{\text{global}} - I_{\text{LOCC}}$$



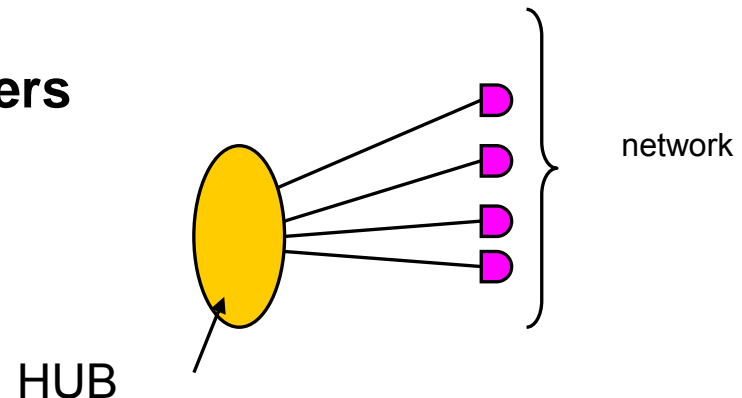
# Introduction

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### 3) **Network of Quantum Computers**

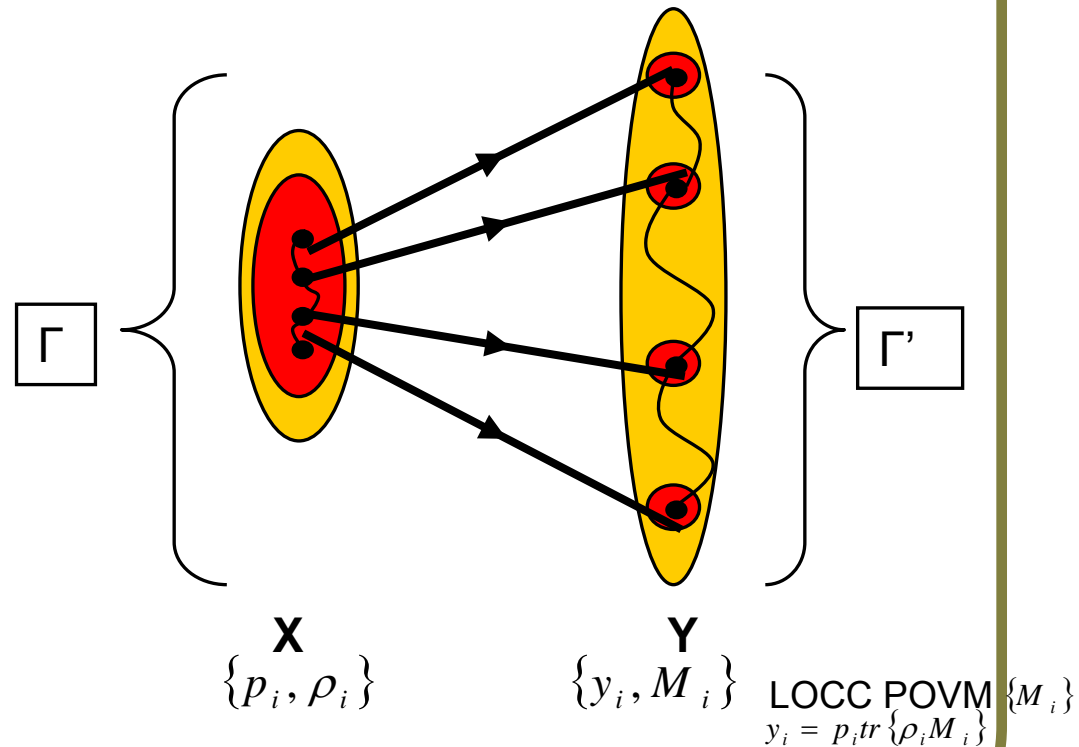


# Introduction

## What is “Classical Capacity of an LOCC Quantum Channel”?

- Encode message  $\Gamma$  into quantum states  $\{p_i, \rho_i\}$
- Decode via LOCC POVM  $\{M_i\}$  to retrieve message  $\Gamma'$

**Capacity** is the optimum rate of message transfer, per use of the channel.



# Introduction

More rigorously...

*n*-th use of the channel

$$\Gamma = (i_1, i_2, i_3, \dots, i_n)$$

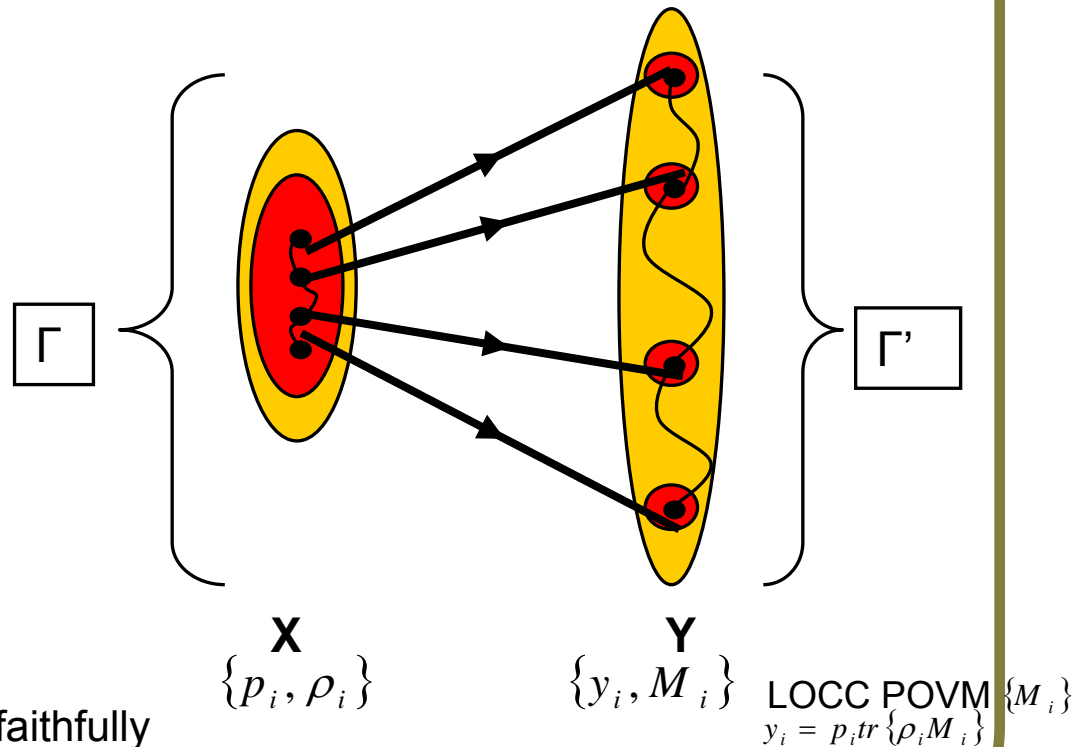


$$\rho_\Gamma = \rho_{i_1}^1 \otimes \rho_{i_2}^2 \otimes \rho_{i_3}^3 \otimes \dots \otimes \rho_{i_n}^n$$

$N$  = number of different possible  $\Gamma$

$$\text{Rate} = \frac{1}{n} \log N$$

Capacity = Maximum rate such that  
message can be decoded faithfully  
(i.e.  $\Gamma$  suff. close to  $\Gamma^*$ )



# Results

## Channel Capacity

$$\rho_{\Gamma} = \rho_{i_1}^1 \otimes \rho_{i_2}^2 \otimes \rho_{i_3}^3 \otimes \dots \otimes \rho_{i_n}^n$$

$$C \leq \log_2 D - \lim_{n \rightarrow \infty} \overline{G(\rho_{\Gamma})} / n$$

a) If entanglement follows superadditivity  $\forall \rho_{\Gamma}, G(\rho_{\Gamma}) \geq nG_{\min}$

$$C \leq \log_2 D - G_{\min}$$

$G(\rho_i)$  geometric measure (entanglement)

$D$  total dimension

## Accessible information

$$I(X : Y) \leq \log_2 D - \min_i G(\rho_i) + \log_2 p_{\max} + H(X)$$

b) If message completely random

$$I(X : Y) \leq \log_2 D - \min_i G(\rho_i)$$

# Proof: channel capacity

## Channel Capacity

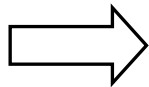
$$\rho_{\Gamma} = \rho_{i_1}^1 \otimes \rho_{i_2}^2 \otimes \rho_{i_3}^3 \otimes \dots \otimes \rho_{i_n}^n$$

## Two steps

- (1) Number of “different” messages = number of states  $N$  that can be discriminated with probability  $\geq \chi$  is bounded by

$$N \leq \frac{D}{\chi^{2^G}}$$

- (2) Take limits



$$C \leq \log_2 D - G_{\min}$$

# Proof: Capacity, (1) bounds on N

To discriminate  $\{\rho_i\}_{i=1..N}$ , the POVM  $\{M_i\}$

$D$  = total dimension

$$I) \quad \sum_i M_i = \mathbf{I}$$

$$II) \quad \mathbf{I} \geq M_i \geq 0$$

$$III) \quad \text{Tr}(M_i \rho_i) = q_i \geq \chi$$

$$VI)* \quad M_i \in \text{SEP}$$

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connection to states  $M_i = s_i \omega_i$

$1 \geq s_i \geq 0$       Density matrix

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connection to states  $M_i = s_i \omega_i$ ,  $s_i = \text{Tr}(M_i)$

$$III) \Rightarrow s_i = \frac{p_i}{\text{Tr}(\omega_i \rho_i)} \geq \frac{\chi}{\text{Tr}(\omega_i \rho_i)}$$

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$$\text{iv) } \omega_i \in \text{SEP}$$

Now the *geometric measure*  $G(\rho_i) := \log_2 \min \frac{1}{\text{Tr}(\omega_i \rho_i)}$  s.t.

$$\text{i) } \omega_i \in \text{SEP}$$

Entanglement monotone  
on pure states

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$$iv) \quad \omega_i \in \text{SEP}$$

$$s_i \geq \chi 2^{G(\rho_i)}$$

Now the *geometric measure*  $G(\rho_i) := \log_2 \min \frac{1}{\text{Tr}(\omega_i \rho_i)}$

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 D = \sum_{i=1}^N S_i \geq \chi \sum_{i=1}^N 2^{G(\rho_i)} \\
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$$\chi \sum_{i=1}^N 2^{G(\rho_i)} \leq D$$

$$G(\rho_i) := \log_2 \min \frac{1}{\text{Tr}(\omega_i \rho_i)} \quad \text{s.t.}$$

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# Proof: Capacity, (1) bounds on N

- i)  $\sum_i s_i \omega_i = \mathbf{I}$
- ii)  $\mathbf{I} \geq s_i \omega_i \geq 0$
- iii)  $s_i = \frac{q_i}{\text{Tr}(\omega_i \rho_i)} \geq \frac{\chi}{\text{Tr}(\omega_i \rho_i)}$
- iv)  $\omega_i \in \text{SEP}$



NECESSARY CONDITION:

To discriminate the states  $\{\rho_i\}_{i=1..N}$  by LOCC with probability of success  $\geq \chi$

$$N \leq \frac{D}{\overline{\chi 2^{G(\rho_i)}}}$$

Where  $\overline{x_i} := \frac{1}{N} \sum_i^N x_i$  is the "average"

# Proof: Capacity, (2) taking limits

## Definition recap

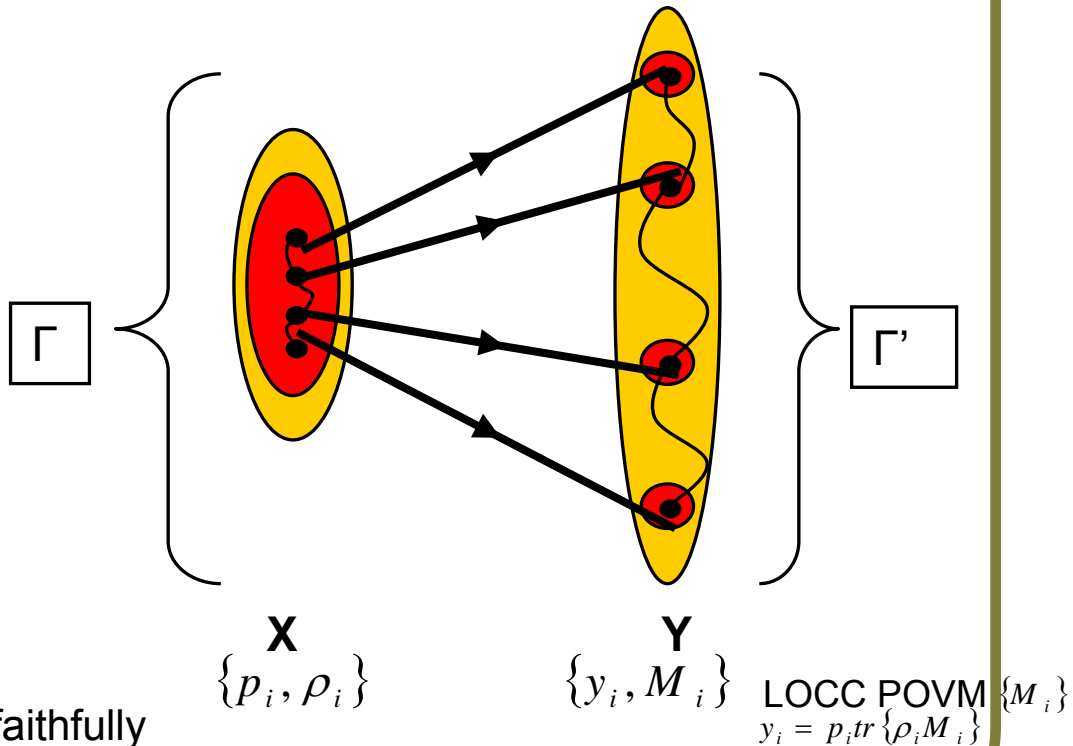
$n$ -th use of the channel

$$\Gamma = (i_1, i_2, i_3, \dots, i_n)$$



$$\rho_\Gamma = \rho_{i_1}^1 \otimes \rho_{i_2}^2 \otimes \rho_{i_3}^3 \otimes \dots \otimes \rho_{i_n}^n$$

$N$  = number of different possible  $\Gamma$



Rate  $R = \frac{1}{n} \log N$

Capacity = Maximum rate such that message can be decoded faithfully (i.e.  $\Gamma$  suff. close to  $\Gamma^*$ )

# Proof: Capacity, (2) taking limits

$$\rho_{\Gamma} = \rho_{i_1}^1 \otimes \rho_{i_2}^2 \otimes \rho_{i_3}^3 \otimes \dots \otimes \rho_{i_n}^n$$

$$N \leq \frac{D}{\chi 2^{\overline{G(\rho_i)}}}$$

$$D = d^n$$

For fixed  $n$  and error  $\chi$

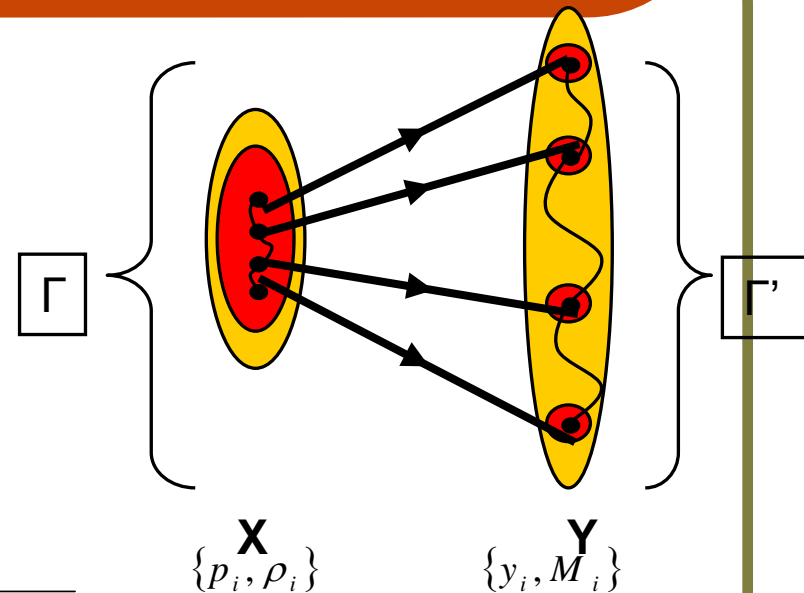
$$R_{n,\chi} = \frac{1}{n} \log N \leq \frac{1}{n} \log \left( \frac{D}{\chi 2^{\overline{G(\rho_i)}}} \right)$$

$$= \log d - \frac{1}{n} \log \chi - \frac{\overline{G(\rho_i)}}{n}$$

Take limits  $n \rightarrow \infty$

$\chi \rightarrow 1$

$$C \leq \log d - \lim_{n \rightarrow \infty} \frac{\overline{G(\rho_i)}}{n}$$



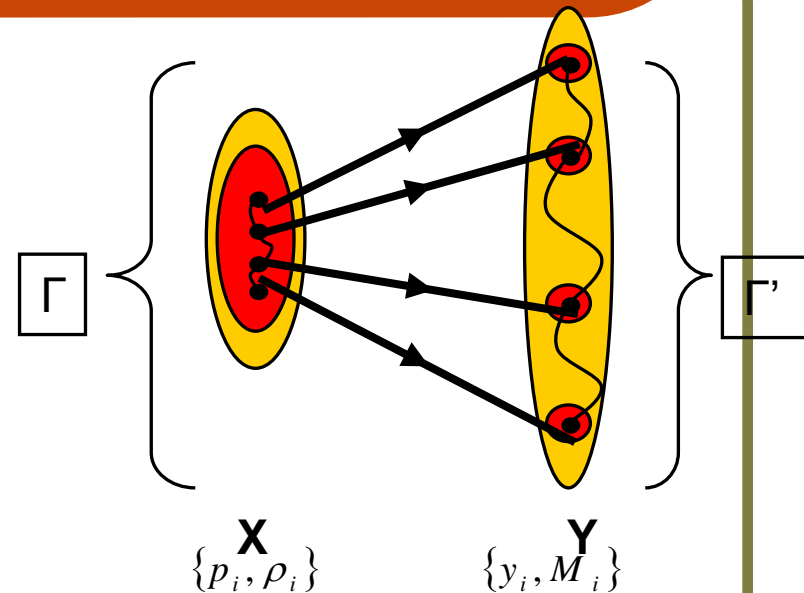
# Proof: Capacity, (2) taking limits

Bound on classical capacity of LOCC quantum channel:

$$C \leq \log d - \lim_{n \rightarrow \infty} \frac{G(\rho_i)}{n}$$

- if  $\forall \rho_{\Gamma}, G(\rho_{\Gamma}) \geq nG_{\min}$

$$C \leq \log_2 D - G_{\min}$$



# Results

## Channel Capacity

$$\rho_{\Gamma} = \rho_{i_1}^1 \otimes \rho_{i_2}^2 \otimes \rho_{i_3}^3 \otimes \dots \otimes \rho_{i_n}^n$$

$$C \leq \log_2 D - \lim_{n \rightarrow \infty} \overline{G(\rho_{\Gamma})} / n$$

a) if  $\forall \rho_{\Gamma}, G(\rho_{\Gamma}) \geq nG_{\min}$

$$C \leq \log_2 D - G_{\min}$$

## Accessible information

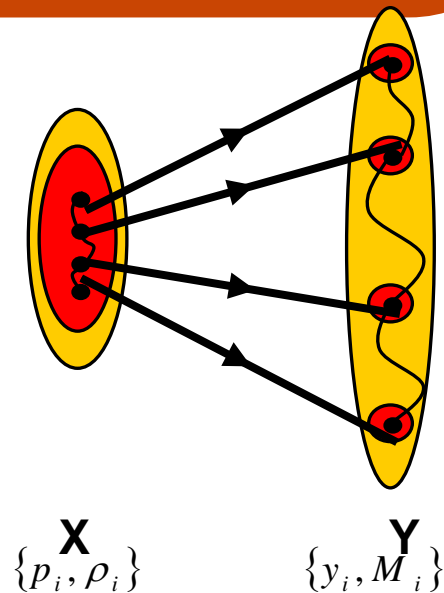
$$I(X : Y) \leq \log_2 D - \min_i G(\rho_i) + \log_2 p_{\max} + H(X)$$

b) If message completely random

$$I(X : Y) \leq \log_2 D - \min_i G(\rho_i)$$

# Proof: Accessible information

- Accessible information is quantity defined between two probability distributions,  $X$  and  $Y$
- Idea for proof:
  - $Y$  depends on optimal measurement
  - our bounds puts limit on max prob that  $X=Y$
  - use to bound accessible information



# Proof: Accessible information

Recall:

$$\text{i) } \sum_i s_i \omega_i = \mathbf{I}$$

$$\text{ii) } \mathbf{I} \geq s_i \omega_i \geq 0$$

$$\text{iii) } s_i = \frac{q_i}{\text{Tr}(\omega_i \rho_i)} \geq \frac{\chi}{\text{Tr}(\omega_i \rho_i)}$$

$$\text{iv) } \omega_i \in \text{SEP}$$

$$\sum_{i=1}^N q_i 2^{G(\rho_i)} \leq D$$

$$G(\rho_i) := \log_2 \min \frac{1}{\text{Tr}(\omega_i \rho_i)} \quad \text{s.t.}$$

$$\text{i) } \omega_i \in \text{SEP}$$

# Proof: Accessible information

- To discriminate  $\{p_i, \rho_i\}_{i=1}^N$  by LOCC, probabilistically, where  $q_i = \text{Tr}(M_i \rho_i)$ , the following bound must hold

$$\sum_{i=1}^N q_i 2^{G(\rho_i)} \leq D$$

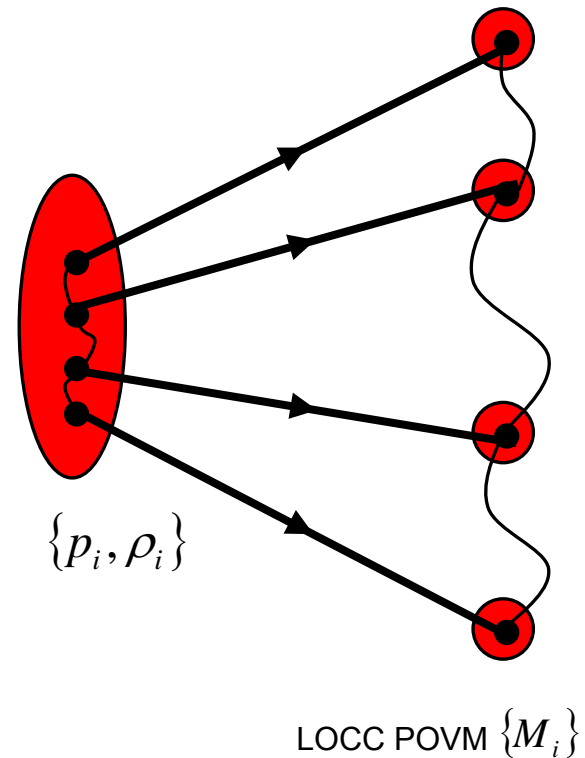
- The probability of successful discrimination

$$P_S = \sum p_i q_i$$

is then bounded by

$$P_S \leq p_{\max} \frac{D}{2^{\min G(\rho_i)}}$$

$q_{\max}$  is the maximum a priori probability



# Proof: Accessible information

- The accessible information is bounded by the classical information theoretic result\*

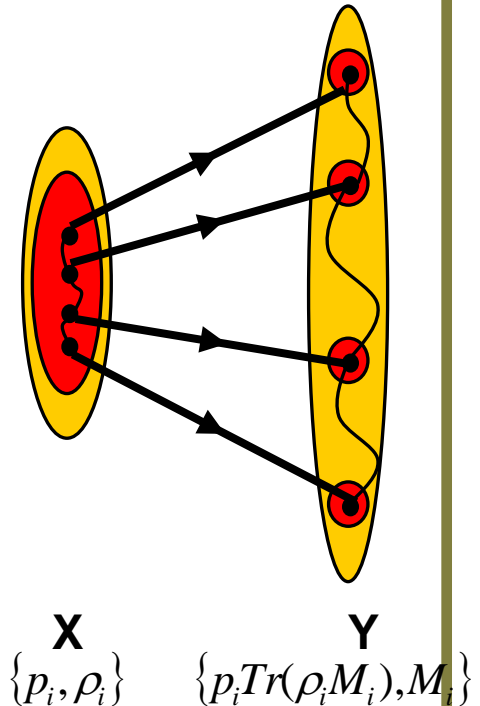
$$I(X : Y) \leq H(X) + \log_2 p_s$$

- X and Y are just probability distributions

Using  $P_S = \sum p_i q_i$        $P_S \leq p_{\max} \frac{D}{2^{\min G(\rho_i)}}$

- Encoding a message into states  $\{p_i, \rho_i\}$ , the accessible information, using only LOCC measurements, is bounded by

$$I(X : Y) \leq \log_2 D - \min_i G(\rho_i) + \log_2 p_{\max} + H(X)$$



\* DiVincenzo, Leung, Terhal, IEEE **48**, 580, (2002)

# Discussion: Compare to bipartite case

- In PRL 91, 117901 (2003)\*, Badziag et al found the following bound for the bipartite case

$$I(X : Y) \leq \log_2 D - \bar{E}_F(\rho_i)$$

(in fact valid for any asymptotic entanglement measure)

- Ours is weaker for pure states, but may be tighter for mixed.
- Suspect we should be able to find similar form to bipartite case

$$I(X : Y) \leq \log_2 D - \min_i G(\rho_i) + \underbrace{\log_2 p_{\max} + H(X)}_{\substack{\text{?} \\ 0}}$$

$\downarrow$   
 $G(\rho_i)$

\* P. Badziag, M. Horodecki, A. Sen and U. Sen PRL 91, 117901 (2003)

# Discussion: Why “distance like” measures?

- In bipartite case,  $E_F$  or  $E_D$ , why now distance like measures  $G$ ,  $E_R$ ?
  - many measures / types of entanglement in multiparty case
  - entanglement distillation / formation, can be seen as state/space discrimination protocols
- Look back at  $d(\rho) := \min \text{Tr}(M)$   
s.t.  $\mathbf{I} \geq M \geq 0, M \in \text{SEP}, \text{Tr}(M\rho) = 1$

$$\frac{d(\rho)}{D} \geq \frac{2^{G(\rho)}}{D}$$

gives minimum probability of error to discriminate  $\rho$  from  $\mathbf{I}$  by a SEP POVM  
(if we insist, if we guess  $\mathbf{I}$  definitely *not*  $\rho$ )

# Discussion: Comments on additivity

- Have some intuition as to why both formation/distillation *and* distance like measures make sense to accessible information / channel capacity
- Super additivity conjectured by many for  $E_f$  and has implications for additivity of channel caps (and related probs)
- We know that dist measures, by definition sub additive and have examples where not additive.

*Normally as a resource, we would expect entanglement to be superadditive, but in context of state discrimination subadditivity may make sense – the more states you have the easier it can be to discriminate them - so they loose their power.*

# Discussion: usefulness of result...

## Why access information by LOCC?

Could be many uses, here are a few

- 1) Environment(s) assisted channel capacity
  - Measures to environment of limited help
- 2) Distribution of secret (secret sharing/ data hiding)
  - gives bound to “secret information”, but no guarantee we can use it..

$$I_{\text{secret}} := I_{\text{global}} - I_{\text{LOCC}} \geq E$$

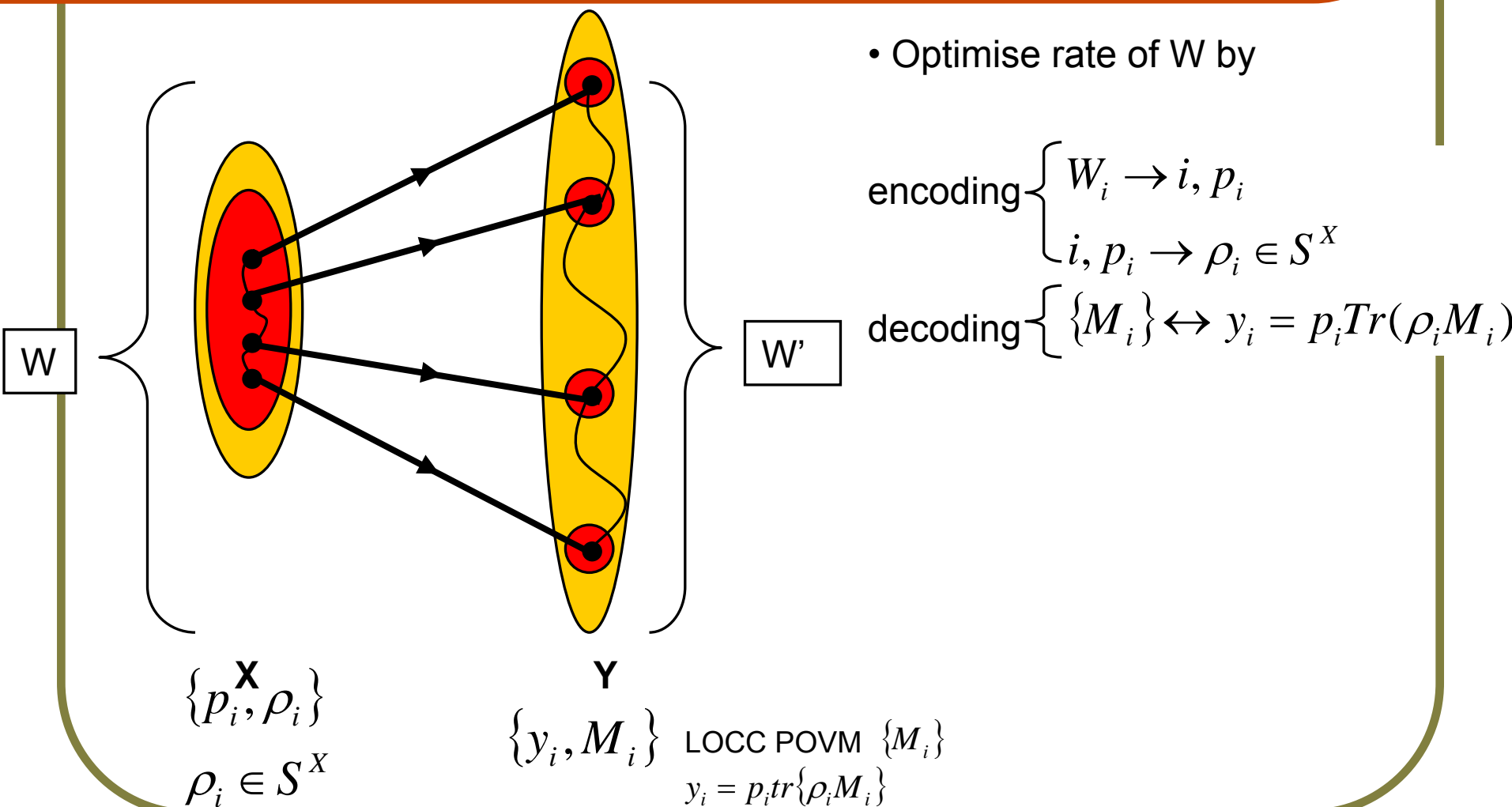
- 3) Network of Quantum Computer
  - Limits amount that can be shared, if entangled

# Conclusions

- Distance like measures and optimisation of LOCC POVM similar problems, enough to give bounds!
- Gives relation to information bounds  $I(X:Y)$ ,  $C$
- Interpretation of entanglement as guaranteeing difficulty in discrimination/ local access of information.
- Although not presented, have examples achieving the bound.
- Future work:
  - uses for protocols, data hiding, error correction...
  - relationship to entanglement *distillation* and manipulation
  - additivity questions

$$I_{secret} := I_{global} - I_{LOCC} \geq E$$

# Appendix: Local channel capacity



# Appendix: Entanglement

Two approaches:           -Physical / operational  
                                  -Abstract

## Physical / Operational

- Usefulness to quantum information tasks

**Bipartite case:** unit of 'usefulness' is the singlet

<i>Entanglement of distillation</i>	$E_D$	number of singlets we can distil from a state by LOCC
<i>Entanglement of formation</i>	$E_F$	number of singlets needed to create state LOCC

**Multiparty case:**

....

**PROBLEM!**

– what are the units of usefulness?

- inequivalent TYPES of entanglement, e.g. W, GHZ

# Appendix: Entanglement

## Abstract

- Satisfying axioms of entanglement -LOCC monotones\*

$$\rho \xrightarrow{\text{LOCC}} \{p_i, \rho_i\} \quad E(\rho) \geq \sum p_i E(\rho_i)$$

E.g. Distance like measures

Global Robustness of Entanglement<sup>§</sup>

$$\mathbf{R}_G(\rho) := \min t$$

$$\text{s.t. } \frac{1}{(1+t)}(\rho + t\omega) \in \text{sep}$$

Relative Entropy of Entanglement\*\*

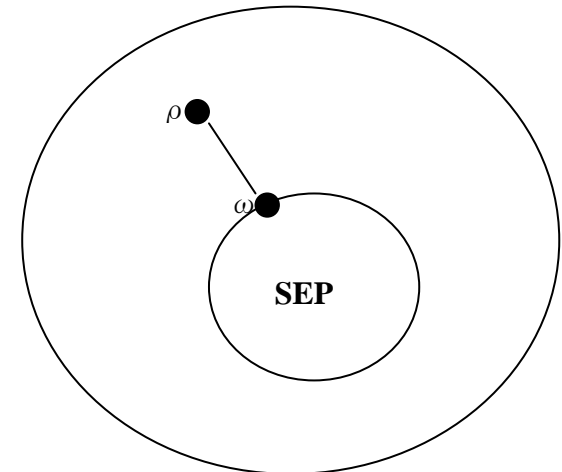
$$E_R(\rho) := \min_{\omega \in \text{sep}} S(\rho \parallel \omega)$$

$$S(\rho \parallel \omega) := \text{tr}(\rho(\log_2 \rho - \log_2 \omega))$$

Geometric Measure<sup>+</sup>

$$E_G(|\varphi\rangle) := -\log_2 \max_{\phi \in \text{prod}} |\langle \varphi | \phi \rangle|^2$$

- Well defined for multipartite case!



But what do they mean?

-here we make the connection to local state discrimination..

\*G. Vidal, J. Mod. Opt. 47(355), 2000.

+ Wie, Ericsson, Goldbart, Munro, Quant Inf. Compu, 4, 252, (2004)

EVidal, Tarrach, PRA 59, 141-155 (1999)

\*\* Vedral, Plenio, PRA 57, 1619 (1998)