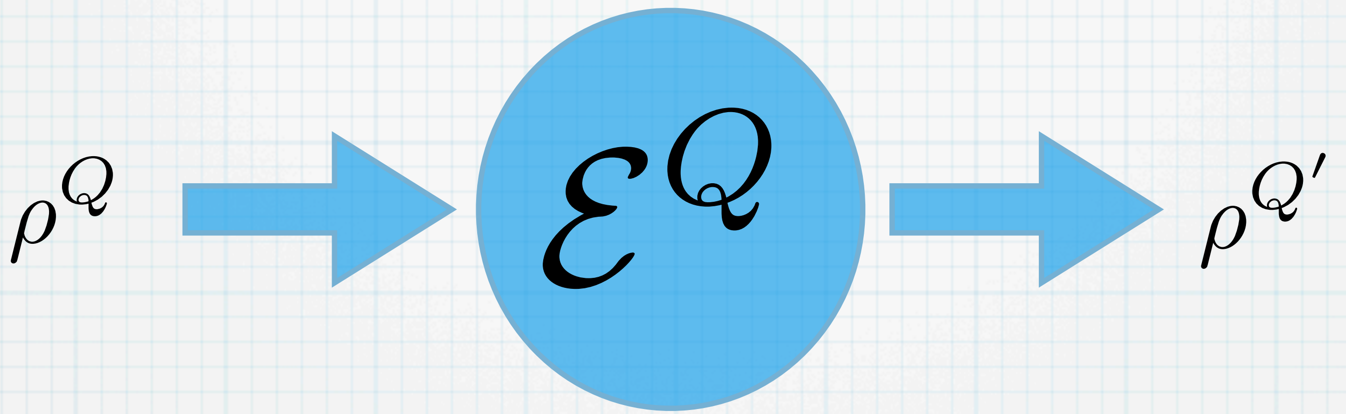
How much entanglement is lost along a channel?

Francesco Buscemi,
TQC 2008, 東京大学
30 Jan 2008

Overview

* Review of some known results about approximate quantum error correction
* Generalization of the theory to other entanglement measures
* Interesting by-product: new sufficient condition for distillability in terms of entanglement of formation

What is "noise"?



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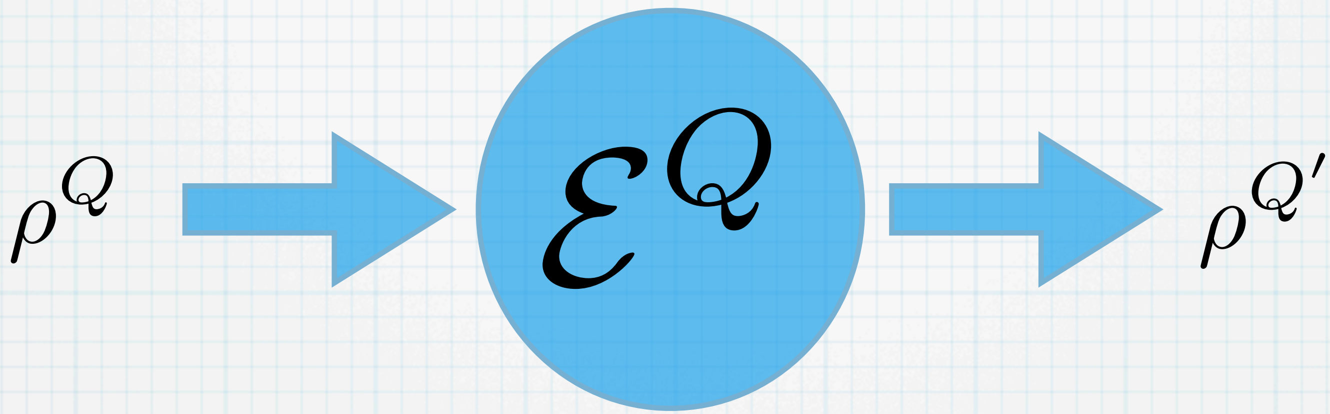

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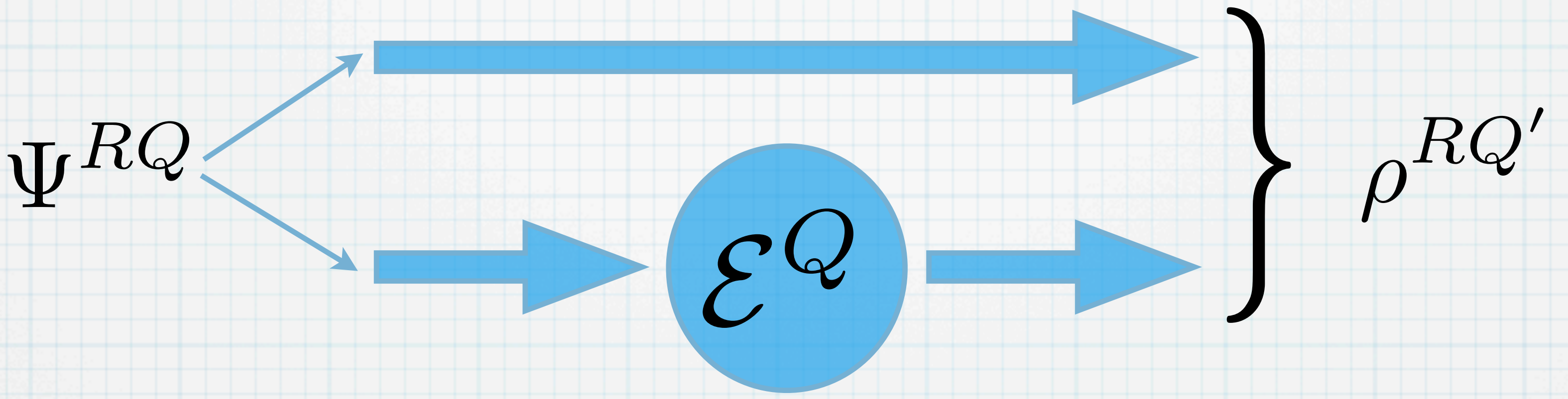
Then we say that, a quantum channel is **noisy on a subspace**, if it cannot be inverted with fidelity 1 and probability 1 on such a subspace.

Superpositions and **entanglement** have to be perfectly recoverable! (cf. decoherence processes, classically noiseless but non recoverable)

How to measure "noisiness"?

Quantum information theory suggests us to purify the input, letting the noise act on one branch of the purification:



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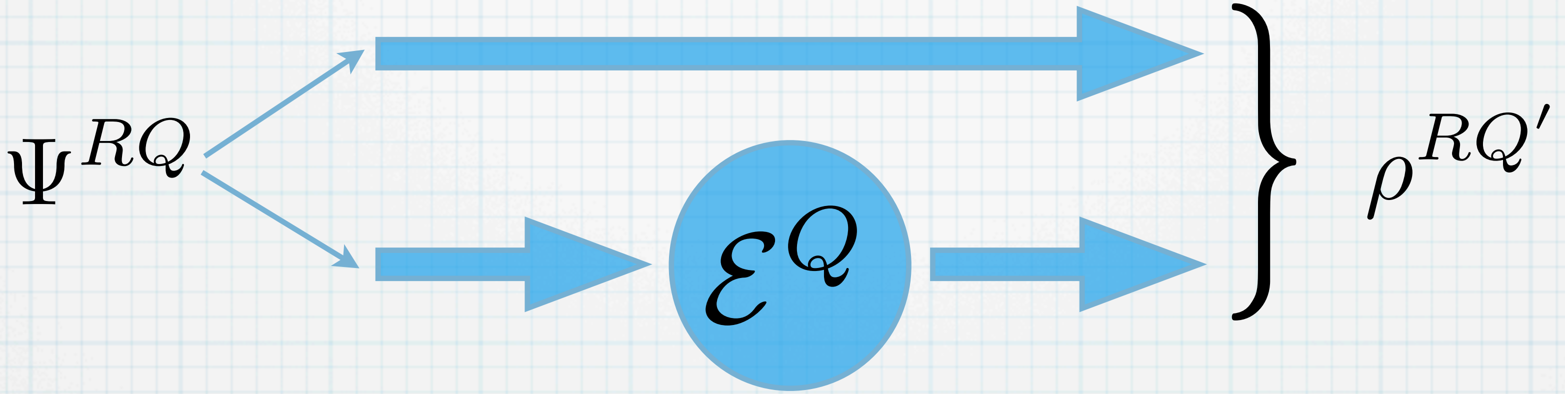

Then we compare the pure bipartite input with the mixed bipartite output by calculating the so-called **entanglement fidelity**

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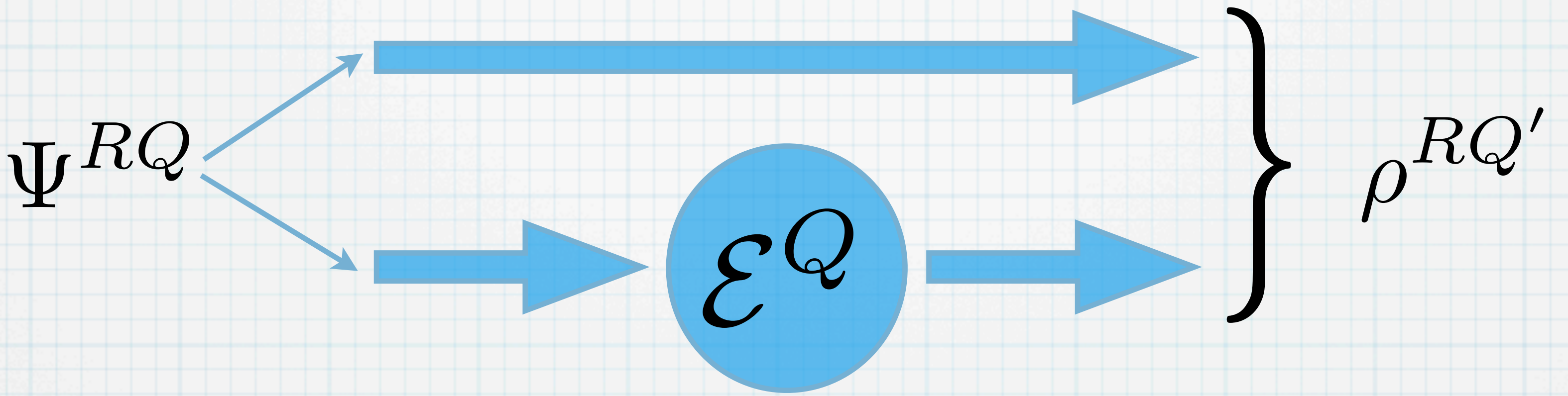

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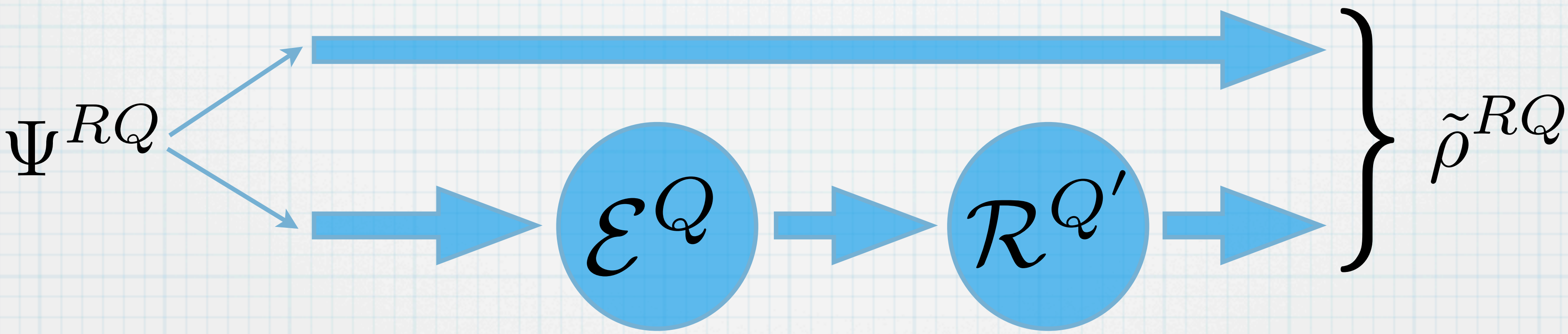
$$F_e(\rho^Q, \mathcal{E}^Q) := \langle \Psi^{RQ} | \rho^{RQ'} | \Psi^{RQ} \rangle$$

This quantity however does not tell us anything about noisiness, as related to irreversibility: it only quantifies how close the given channel $\mathcal{E}^Q$ is with respect to the identity channel id^Q

Approximate correction

So, how noisy is a given channel? By using the idea of entanglement fidelity, we can define the **corrected entanglement fidelity** as

$$F_c(\rho^Q, \mathcal{E}^Q) := \max_{\mathcal{R}^{Q'}} \langle \Psi^{RQ} | \tilde{\rho}^{RQ} | \Psi^{RQ} \rangle$$



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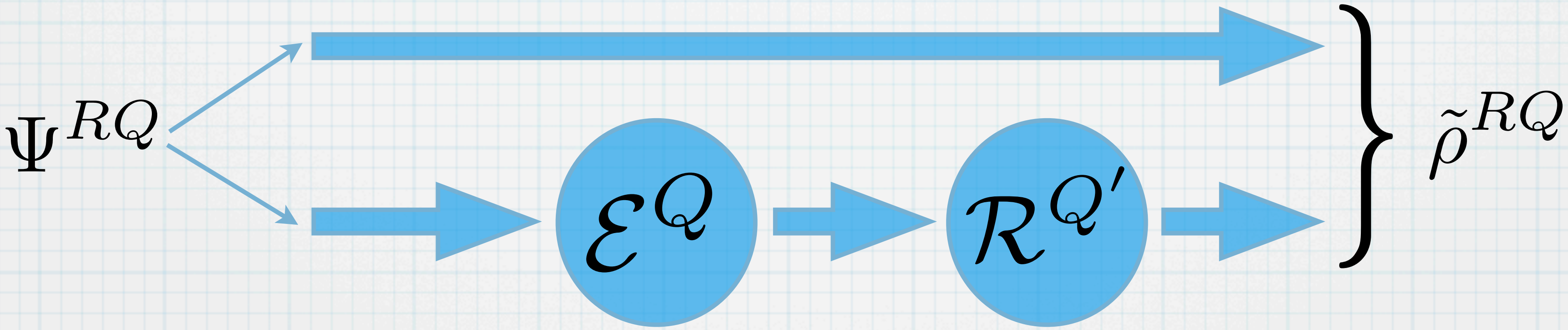
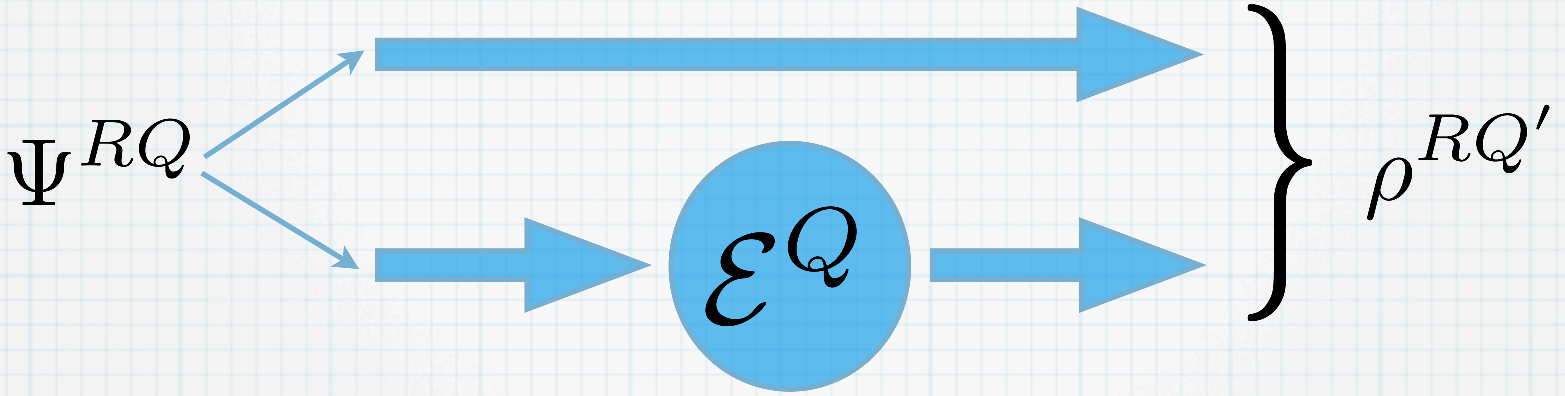


Still, the maximization over all possible corrections is not handy at all... Can we overcome this technical problem?

Coherent information loss



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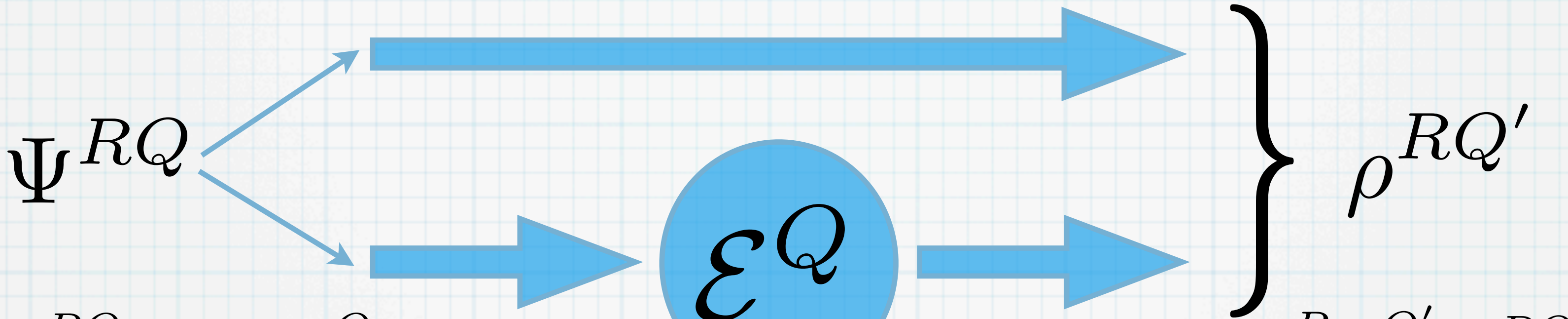

$I_{coh}^{R\to Q}(\Psi^{RQ}) = S(\rho^Q)$

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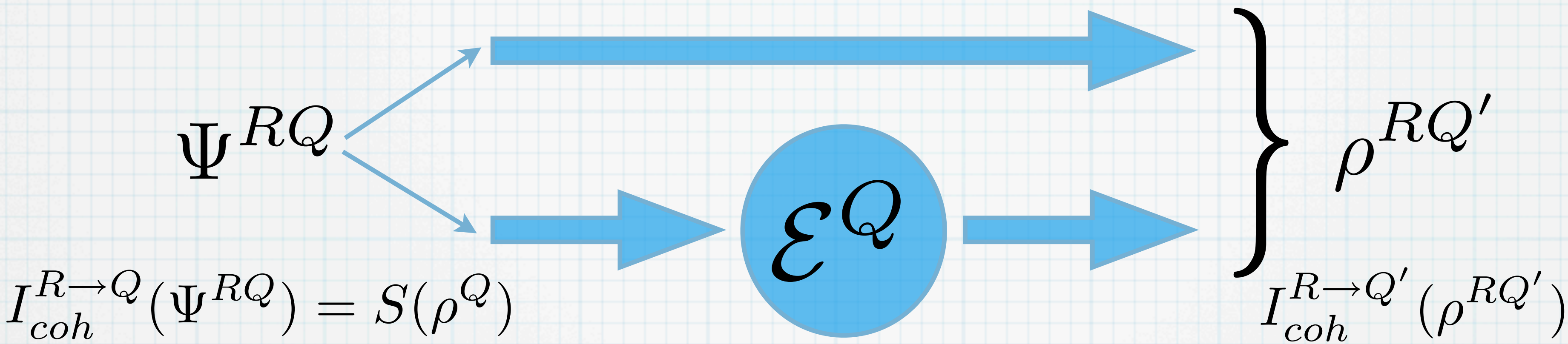


$$I_{coh}^{A\to B}(\rho^{AB}) := S(\rho^B) - S(\rho^{AB})$$

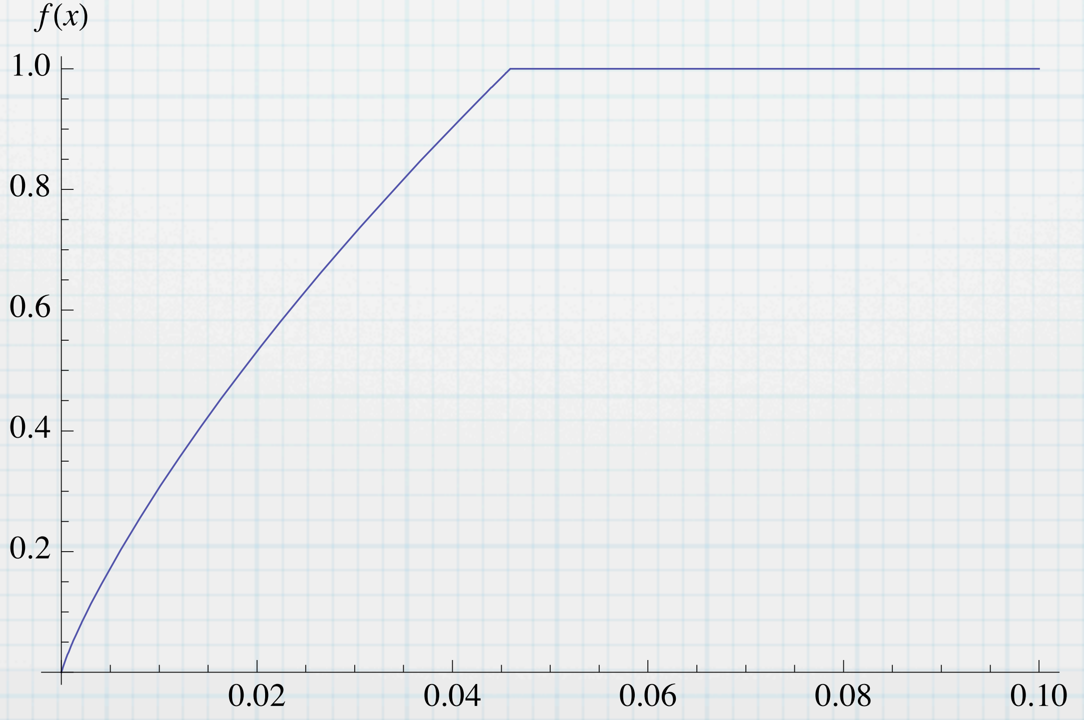
The following bound, due to Schumacher and Westmoreland (2002), is exactly what we are searching for: Let $\delta_{coh}(\rho^{RQ'}) := S(\rho^Q) - I_{coh}^{R\to Q'}(\rho^{RQ'})$ be the **coherent information loss**; then there exists a correction such that

$$F_c(\rho^Q, \mathcal{E}^Q) \geq 1 - \sqrt{2\delta_{coh}(\rho^{RQ'})}$$

Opposite direction: quantum Fano inequality

On the other hand, we have that the converse is also true

$$\delta_{coh}(\rho^{RQ'}) \leq \mathrm{f}\left(1 - F_c(\rho^Q, \mathcal{E}^Q)\right)$$



What is coherent information?

* It is NOT an entanglement monotone.
* If $I_{coh}^{A\to B}(\rho^{AB}) > 0$ then the state is entangled; if the state is separable $I_{coh}^{A\to B}(\rho^{AB}) \leq 0$.
* It is related with the quantum capacity of channels.
* It is a lower bound on the one-way distillable entanglement (hashing ineq.) $E_D^{A\to B}(\rho^{AB}) \geq I_{coh}^{A\to B}(\rho^{AB})$

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A channel is approximately invertible if and only if the loss of coherent information is small. What's special about coherent information? For pure states, it is an entanglement measure. Why a generalization to other entanglement measures is not straightforward?

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The reason lies at the roots of entanglement theory...

Very quickly: some entanglement measures

$$\max\{I_{coh}^{A\to B}(\rho^{AB}),0\} \leq E_D^{A\to B}(\rho^{AB}) \leq E_\bullet(\rho^{AB}) \leq E_F(\rho^{AB}) \leq \min\{S(\rho^A),S(\rho^B)\}$$

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Notice: we are considering entanglement measures under LOCC restriction (distant laboratories paradigm)

Typical behavior

* for pure states, measures coincide

$$I_{coh}^{A\to B}(\Psi^{AB}) = S(\rho^B) = S(\rho^A)$$

* for a randomly chosen mixed state, as dimension increases

$$E_D^{A\to B}(\rho^{AB}) \to 0$$
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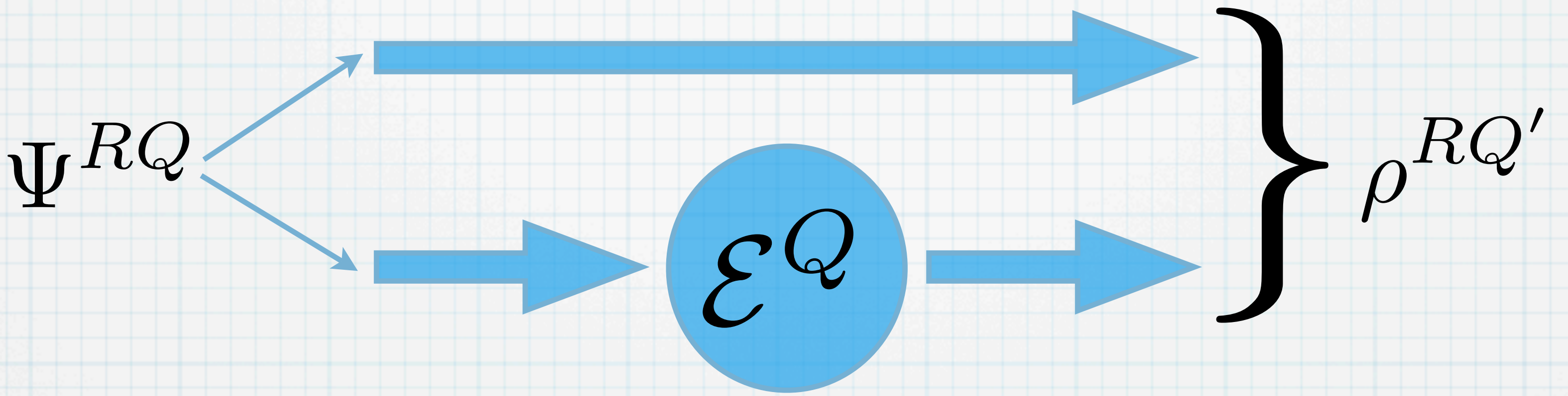
irreversibility gap under LOCC!!

Other loss functions: Entanglement of Formation



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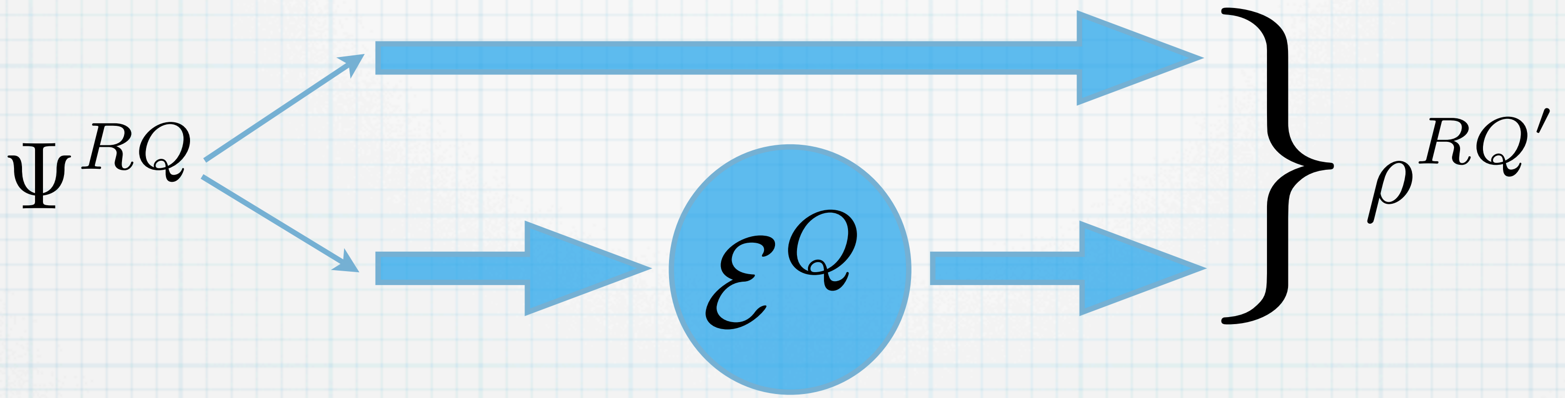

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$$F_c(\rho^Q, \mathcal{E}^Q) \geq 1 - \sqrt{2\delta_{coh}(\rho^{RQ'})}$$

Squashed measures loss

Let us now consider another important threshold for entanglement measures, that is, half of the quantum mutual information. If a measure satisfies

$$E_\bullet(\rho^{AB}) \leq \frac{I^{A:B}(\rho^{AB})}{2}$$

where $I^{A:B}(\rho^{AB}) := S(\rho^A) + S(\rho^B) - S(\rho^{AB})$, then

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satisfied by squashed ent. and distillable ent.

Comparing the bounds

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On the other hand it holds that:

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By putting pieces together

$$\delta_{coh}(\rho^{AB}) \leq \mathrm{f}\left(\sqrt{2(2d_A d_B - 1)^2 \delta_F(\rho^{AB})}\right)$$

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Surprisingly, this is the REAL dimension of the set of bipartite states

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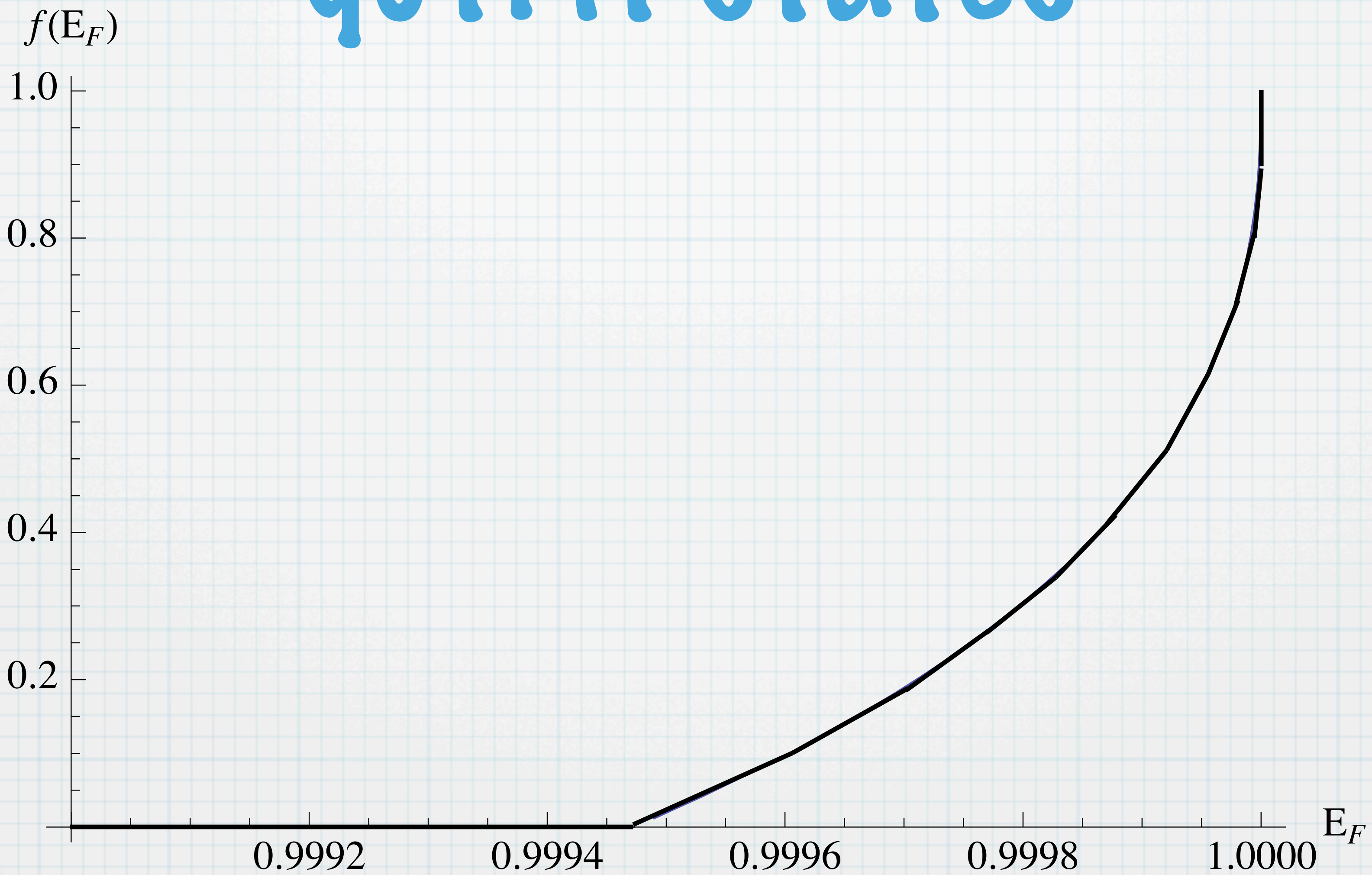
Example: Plot for two-qutrit states



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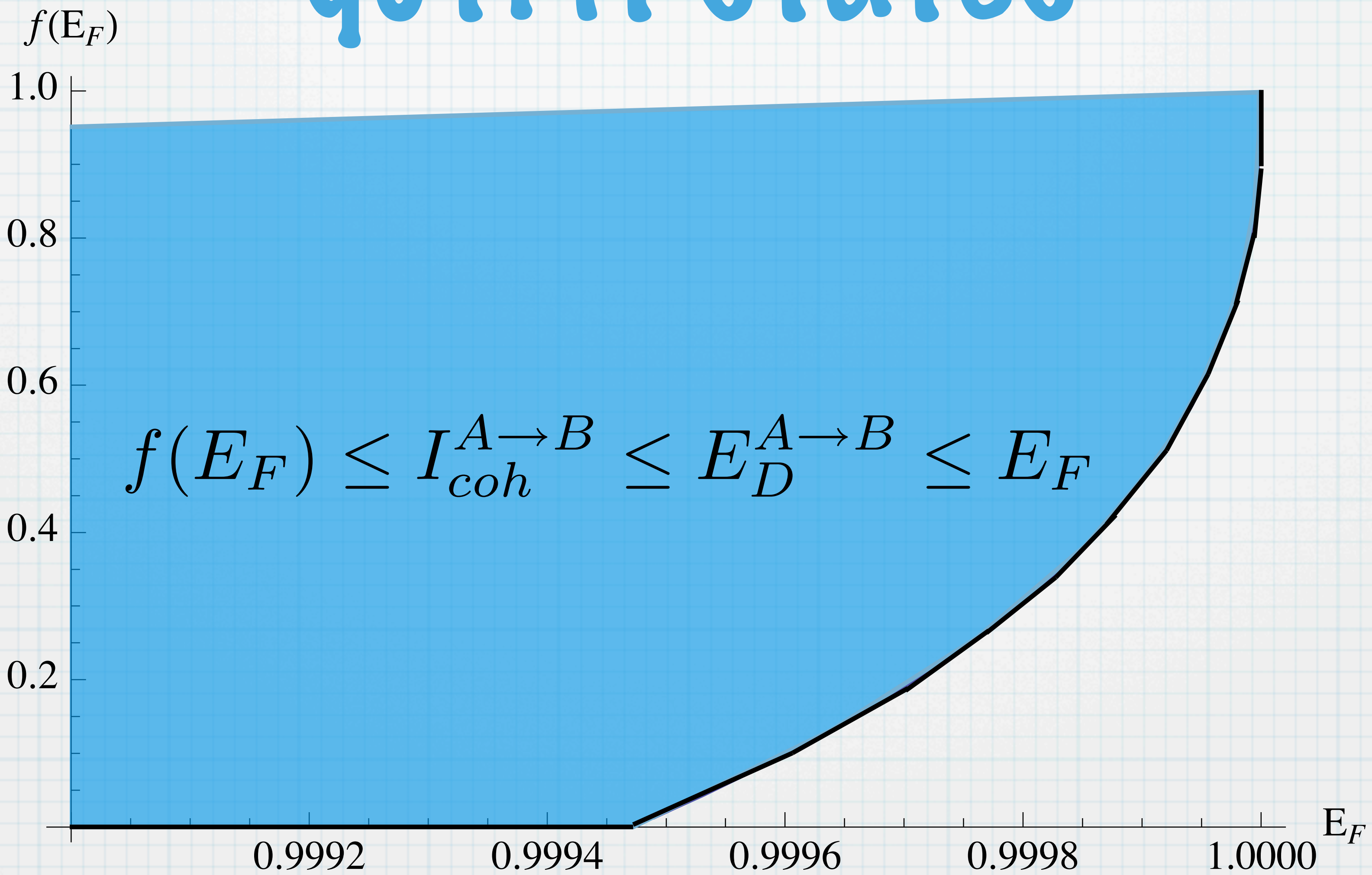

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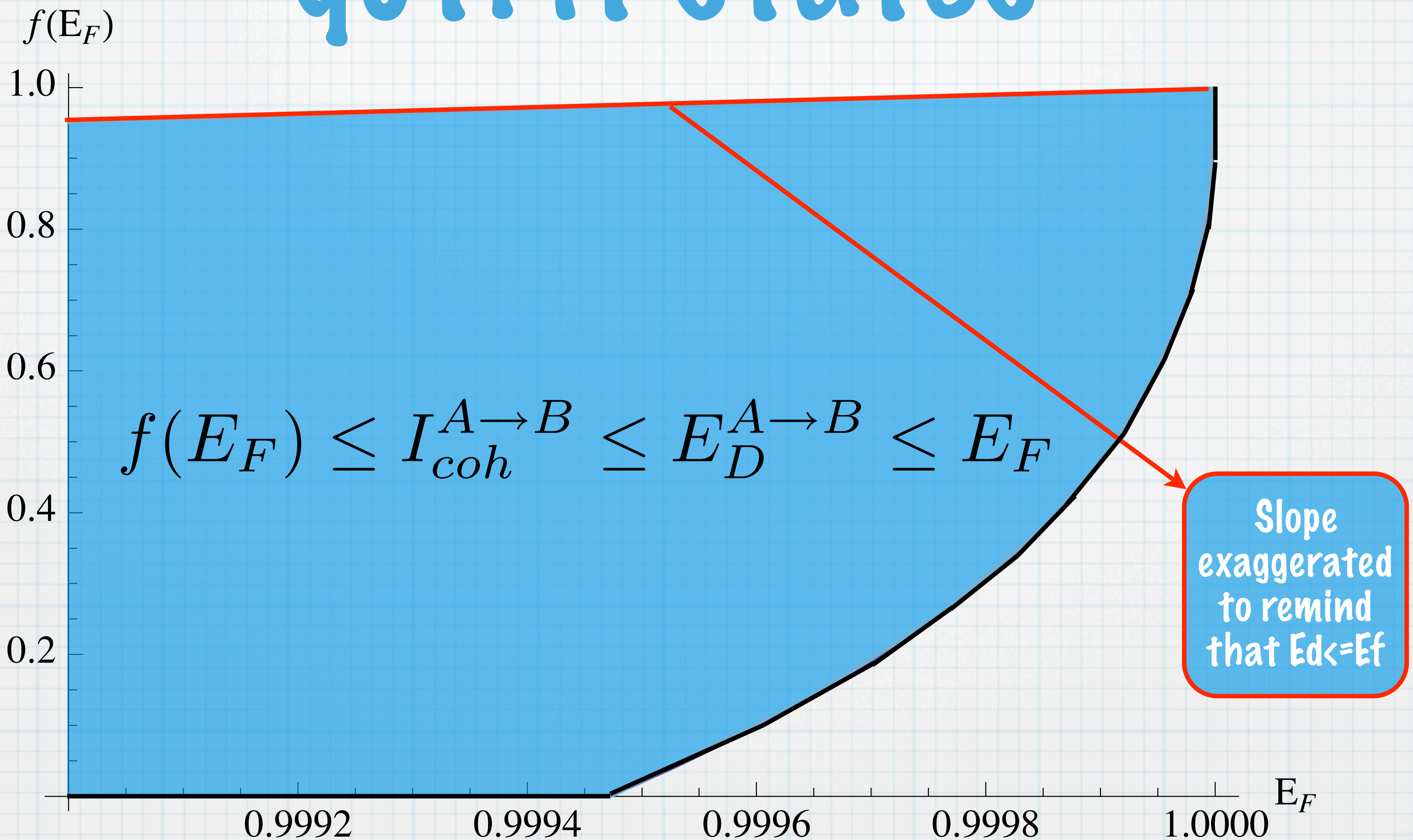

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